1.5 Hydrostatic equation.

Under hydrostatic equilibrium, the local pressure (which is isotropic - i.e., independent of direction) balances weight per unit area. Since weight per unit area due to the mass of air above height z is

$$\int_z^\infty g\rho(z')dz' \ ,$$

hydrostatic equilibrium is

$$p(z) = \int_z^\infty g\rho(z')dz' \qquad (1.16)$$

where $g \approx 9.8\,\mathrm{ms}^{-2}$ is gravitational acceleration. The differential form of this equation is

$$\boxed{\frac{\partial p}{\partial z} = -g\rho = -\frac{gp}{RT}}, \qquad (1.17)$$

which expresses the balance of pressure and gravitational forces across a layer of infinitesimal vertical thickness. We can integrate this equation, using the ideal gas law to eliminate density ρ ,

$$\frac{1}{p}\frac{\partial p}{\partial z} = \frac{\partial \ln p}{\partial z} = -\frac{g}{RT},$$

$$\boxed{p(z) = p(0)\exp\left(-\int_0^z \frac{g}{RT}dz'\right) = p(0)\exp\left(-\int_0^z \frac{dz'}{H}\right)} \qquad (1.18)$$

which expresses pressure decrease with height in terms of the scale height $H \equiv \frac{RT}{g}$. Below approximately 100km, fractional variations in temperature (in Kelvins) are relatively small, mainly $< \pm 20\%$, so H is roughly constant with a typical value

$$H \approx \frac{287\text{x}250}{9.8} \approx 7.3\,\text{km}.$$

This implies that pressure varies approximately exponentially with height,

$$p(z) \approx p(0)\exp\left(-\frac{z}{H}\right) \ , \qquad (1.19)$$

and leads to the useful rule of thumb:

Below 100 km, pressure decreases by one order of magnitude about every 15km of altitude gain.

For example, average surface pressure is about 1000hPa, so pressure at 90km averages about 0.001hPa.

Similarly for density:

$$\rho(z) = \frac{p(z)}{RT} \approx \frac{p(0)}{RT}\exp\left(-\frac{z}{H}\right) \approx \rho(0)\exp\left(-\frac{z}{H}\right), \tag{1.20}$$

so that density also decreases by one order of magnitude about every 15 km of altitude gain.

Another very useful version of the hydrostatic equation is obtained by turning eq. 1.17 upside down and integrating to obtain the hypsometric equation:

$$\boxed{z(p_1) - z(p_0) = \int_{p_1}^{p_0} \frac{RT(p)}{g} d\ln p = \overline{H}\ln\left(\frac{p_0}{p_1}\right)} \;. \tag{1.21}$$

The hypsometric equation expresses the thickness between isobaric layers, $\{z(p_1) - z(p_0)\}$ in terms of ln(pressure) and ln(pressure)-weighted mean scale height $\overline{H}$ between the isobaric surfaces p_0 and p_1. Thus, for example, when a radiosonde measures data for an atmospheric sounding (vertical profile of thermodynamic properties), it measures $T(p)$ directly. From the hypsometric equation, thickness between isobaric layers is inferred for all isobaric layers from the surface to the top of the sounding, and hence the height of all isobaric layers to the top of the sounding is inferred.

1.6 Applications.

Example 1: Show that the number of molecules per unit area in an atmospheric column to the "top" of an isothermal atmosphere, ($\mathcal{N}$), is $n_0 H$ where $n(0)$ is number density at the ground ($z = 0$).

Solution:

$$\mathcal{N} = \int_0^\infty n\,dz = \int_0^\infty \frac{\rho}{m}dz = \int_0^\infty n(0)\exp\left(-\frac{z}{H}\right)dz = n(0)H. \tag{1.22}$$

Example 2: Derive expressions for $p(z)$ and $\rho(z)$ for an atmospheric layer in which the lapse rate, $-\frac{dT}{dz} = \text{constant} \equiv \Gamma$.

Solution: Using the hydrostatic equation,

$$d\ln p = -\frac{g}{RT}dz = -\frac{g}{RT}\left(\frac{dz}{dT}\right)dT = \frac{g}{R\Gamma}\cdot\frac{dT}{T} = \frac{g}{R\Gamma}d\ln T.$$

Integrating and taking antilogs on both sides,

$$\boxed{p(z) = p(0)\left[\frac{T(z)}{T(0)}\right]^{\frac{g}{R\Gamma}} = p(0)\left[1-\frac{\Gamma z}{T(0)}\right]^{\frac{g}{R\Gamma}}} \quad . \qquad (1.23)$$

Density is obtained similarly,

$$\rho(z) = \frac{p(z)}{RT(z)} = \frac{p(0)}{RT(0)}\cdot\frac{\left[1-\frac{\Gamma z}{T(0)}\right]^{\frac{g}{R\Gamma}}}{\left[1-\frac{\Gamma z}{T(0)}\right]} = \rho(0)\left[1-\frac{\Gamma z}{T(0)}\right]^{\frac{g}{R\Gamma}-1}, \qquad (1.24)$$

or

$$\rho(z) = \rho(0)\left[\frac{p(z)}{p(0)}\right]^{1-\frac{R\Gamma}{g}} \quad . \qquad (1.25)$$

These analytic expressions are especially useful for the construction of "standard atmospheres" since any temperature profile can be approximated to the degree of precision required by a continuous profile consisting of an arbitrary number of layers each of which has a constant lapse rate.

<u>Example 3</u>: What is the difference in height of the 500hPa surface between Oakland and Seattle if the mean temperature of the 1000 to 500hPa layer is 270K at Oakland and 260K at Seattle, and if the height of the 1000hPa surface is the same at both places?

<u>Solution</u>: Use the hypsometric equation,

$$z(500\text{hPa}) - z(1000\text{hPa}) = \int_{500}^{1000}\frac{RT}{g}d\ln p = \frac{R\overline{T}}{g}\ln 2 = 20.3\overline{T},$$

so thicknesses are 5481m at Oakland and 5278m at Seattle. The height difference is 203 meters sloping downward toward the north.

<u>Example 4</u>: The Atmospheric Sciences building is 20m tall. On a good aneroid barometer, how much pressure change would you see if you carried the barometer from bottom to top of the building?

Solution: Because the height change is small, we can conveniently approximate the hypsometric equation,

$$\boxed{d\ln p \approx \frac{\delta p}{p} \approx \frac{\delta z}{H}}, \qquad H \approx \frac{287\text{ms}^{-2}\text{x}287\text{K}}{9.8\text{ms}^{-2}} = 8400\text{m},$$

$$\delta p \approx \frac{(1000\text{hPa})\text{x}(20\text{m})}{8400\text{m}} = 2.4\text{hPa}.$$

Example 5: You fly a light plane from Boeing Field and return. When you leave, your altimeter setting, 29.93", is correct, i.e. it gives an altitude reading of 0 when you are on the runway. When you return, surface pressure has fallen 5hPa, but you fail to reset your altimeter. What does the altimeter read when you hit the runway?

Solution: Since surface pressure is lower upon your return, your altimeter (which is an aneroid barometer) will give an elevation reading above the runway when you are actually on the runway. By how much? Noting that 29.93" corresponds to 1013hPa, and assuming that the scale height is approximately 8400m, the height error Δz is approximately

$$\Delta z \approx H\frac{\Delta p}{p} \approx 42\text{m}, \qquad \text{where } \Delta p \text{ is the pressure change during your flight}.$$

Now suppose that, in addition to the pressure decrease, temperature over the airport has declined by 5°C since you left. When your altimeter reads 1000m during your approach, how much different will your actual altitude be from what it was at the same altimeter reading when you took off?

Solution: In addition to the 42m lower due to surface pressure change, there will be a change due to thickness change. Again, this will be in the "wrong" direction (i.e. the direction of giving you a misleading margin of safety in altitude). The difference is approximately

$$\Delta z \approx \frac{R\Delta T}{g}\frac{\delta z}{H} = \delta z\frac{\Delta T}{T} \approx -1000 \cdot \frac{5}{287} = -17\text{m}$$

where δz=1000m is original layer thickness. So the total altimeter change at 1000m will be the sum of contributions from temperature change and thickness change, or -41 -17 = -59m. Since this is all in the "wrong" direction, the moral is: reset your altimeter! (or use GPS).