1.7 Small corrections.

In the discussion of hydrostatics, two relatively small factors were neglected:

- Spatial variations of g (it was treated as a constant),
- Spatial variations of R due to variations in composition and hence mean molecular weight.

Hydrostatic and hypsometric formulations expressed in terms of the scale height are still correct, but variations in scale height now arise from variations of g and R as well as T. In this section, we describe how effects of those variations are usually handled.

(a) Variations of g. g used in the hydrostatic equations is actually an effective gravity, incorporating more factors than just Newtonian gravitional acceleration toward the center of mass. T is given by

$$\mathbf{g} = \frac{a^2}{(a+z)^2} g_0 \hat{\mathbf{k}}$$

where a is mean radial distance to Earth's surface from the center of mass (mean radius), z is height above the surface, g_0 is standard gravitational acceleration at that level, and $\hat{\mathbf{k}}$ is the radially outward unit vector from the center of the planet. It also includes centrifugal acceleration due to Earth's rotation at the rate Ω, which adds a term

$$\Omega^2 (a+z) \cos\varphi \hat{\mathbf{e}}_{\mathbf{r}}$$

where φ is latitude and $\hat{\mathbf{e}}_{\mathbf{r}}$ is the unit vector radially outward from axis of rotation. In addition, a term due to departures of mass distribution from spherical symmetry due to both centrifugal acceleration and non-spherical compositional inhomogeneities $\varepsilon(\lambda, \varphi, z)$, where λ is longitude, must also be added (see MS, pp. 143-145).

All of these effects can be made invisible by introducing geopotential coordinates whose vertical coordinate surfaces are surfaces of constant gravitational potential Φ,

$$\Phi(\lambda, \varphi, z) = -\int \mathbf{g} \bullet \mathbf{d}\xi \quad , \qquad (1.26)$$

the geopotential relative to some constant apparent gravity potential reference surface. Note that, because of the centrifugal stretching effect contained in apparent gravity, surfaces of constant geopotential on Earth are very slightly flattened at the poles. Jupiter, with rotation period 10.7hr and radius 71,000km has very noticeably flattened surfaces of apparent gravitational potential.

If global mean sea level (msl) corresponds to the reference geopotential surface, then

$$\Phi(z) = \int_0^z g dz \tag{1.27}$$

where $g = |\mathbf{g}|$ and msl is the zero geopotential surface. Now let

$$\Phi\left(z^*\right) = g_0 z^* \tag{1.28}$$

where $g_0 = 9.8\mathrm{ms}^{-2}$ is "standard gravity" and z^* is geopotential height. Now all space dependent variables can be expressed in terms of geopotential height z^* instead of geometric height z. The only price for this shift is that vertical coordinate surfaces do not exactly correspond to geometric height (above msl). For example, near 60km, geopotential surfaces are about 1% lower than the corresponding geometric height surfaces. This is not a problem; transformation to geometric height can always be carried out if needed. Using geopotential height, the hydrostatic equation becomes

$$dp = -\rho g dz = -\rho g_0 dz^* \ . \tag{1.29}$$

Geopotential height will be used henceforth, but for convenience, the asterisk $(\ )^*$ will be dropped, and g will stand for standard gravity g_0.

(b) <u>Variations in composition</u>. Composition variations imply variations in gas constant R due to variations in mean molecular mass. Composition variations large enough to produce significant variations in R occur either because the major species are varying, as they do in the upper atmosphere above about 100km, or because of variations in mixing ratio of water vapor. It is customary to incorporate these variations into an artificial temperature called the scale temperature T_{scale}, keeping R fixed at its dry air value $287\mathrm{m}^2\mathrm{s}^2$ in calculations. Thus, in general, T_{scale} is given by

$$T_{\mathrm{scale}} = \frac{\overline{M_0}}{\overline{M}} T \quad \text{where} \quad \overline{M_0} = 28.97.$$

By far the most important application is to water vapor. Let $e \equiv$ partial pressure of water vapor. Then, since $\frac{e}{p} = \frac{n_w}{n}$,

$$R = \frac{R^*}{\left[M_d\left(1 - \frac{e}{p}\right) + M_w \frac{e}{p}\right]} = \frac{R^*}{M_d}\left[1 - \frac{e}{p}\left(1 - \varepsilon_w\right)\right]^{-1} \tag{1.30}$$

where $\varepsilon_w = \frac{M_w}{M_d} \approx \frac{18}{28.97} = 0.621$. Instead of allowing R to change, we artificially fix it at its standard dry air value $\overline{M_0}$ and introduce the virtual temperature T_v:

$$\boxed{T_v = \frac{T}{\left[1 - \frac{e}{p}\left(1 - \varepsilon_w\right)\right]} \approx \left(1 + 0.61q_v\right)T} \qquad (1.31)$$

where q_v is the mass mixing ratio of water vapor. Then the ideal gas law is:

$$\boxed{p = R\rho T_v} \qquad (1.32)$$

in which the gas constant has the standard dry air value $287 \mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$. Note that moist air is less dense than dry air at the same temperature and pressure because water vapor molecules replace heavier dry air molecules. Under humid conditions, T_v can exceed T by as much as 2%.