3.3 Applications.

Example 1: A dry atmospheric column between 1000 and 500 hPa absorbs heat at the rate of $100 Wm^{-2}$. What is the static thickness change after 8 hours?

Solution: Static change implies that there are no atmospheric motions and the process takes place isobarically. Then

$$\Delta Q = \Delta H = 8(\mathrm{hr})x3600(\mathrm{s/hr})x100\,\mathrm{Wm}^{-2} = \int_{z_{1000}}^{z_{500}} \rho c_p \Delta T dz = \frac{1}{g}\int_{500}^{1000} c_p \Delta T dp$$

$$= \frac{c_p x5x10^4(\mathrm{Pa})}{g}\overline{\Delta T}$$

for the column. Also,

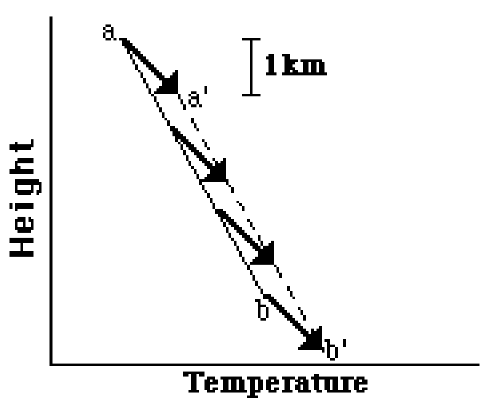
$$\Delta(\text{thickness}) = \frac{R}{g}\overline{\Delta T}\ln\frac{1000}{500} = \frac{R}{g}\frac{\ln\frac{1000}{500}}{5x10^4(\mathrm{Pa})}\Delta Q.$$

$$\overline{\Delta T} = 0.56\mathrm{K},\quad \Delta(\text{thickness}) = 11.4\mathrm{m}\ .$$

This is the approximate magnitude of the thickness changed forced by the daily cycle of solar heating.

Example 2: The lapse rate between 800 and 400 hPa is 7K/km. The entire atmospheric column including this layer subsides (sinks) 1 km in 1 day. If the subsidence is adiabatic, what is the thickness change of the layer?

Solution: The situation is illustrated in the figure below in which the straight line ab represents the layer temperature profile with uniform lapse rate 7K/km, and the heavier arrows represent parcels originating along that profile descending 1km at the dry adiabatic lapse rate 9.8K/km.



As a consequence of the difference between the layer lapse rate and the dry adiabatic lapse rate, the temperature profile of the entire layer shifts to the warmer profile a'b' as shown. The temperature and thickness changes between ab and a'b' are given by

$$\Delta T = \left[\left(\frac{dT}{dz}\right)_{\text{adiabatic}} - \left(\frac{dT}{dz}\right)_{\text{profile}}\right]\Delta z = -\left[\left(\frac{g}{c_p}\right) - 7x10^{-3}(K/km)\right]\Delta z = 2.8K.$$

(Note that Δz is negative.)

$$\Delta(\text{thickness}) = \frac{R}{g}\Delta T \ln\left(\frac{800}{400}\right) = 57m.$$

Note that, in the thickness calculation, we had to assume that the same changes were occurring above the top of the original layer as in the layer itself so that the final change still corresponded to change in the 800-400hPa layer.

A more formal solution may be illuminating. Start with

$$\frac{d\ln\theta}{dt} = \frac{d\ln T}{dt} - \kappa\frac{d\ln p}{dt} = \frac{1}{c_p T}\dot{q}.$$

Since this is a problem in which there is no dependence on horizontal spatial variables, so $T = T(z,t)$ and $p = p(z,t)$ only. Consequently,

$$\frac{d\ln T}{dt} = \frac{1}{T}\left[\frac{\partial T}{\partial t} + \frac{\partial T}{\partial z}\frac{dz}{dt}\right]$$

and

$$\frac{d\ln p}{dt} = \frac{1}{p}\left[\frac{\partial p}{\partial t} + \frac{\partial p}{\partial z}\frac{dz}{dt}\right] \approx 0 + \frac{\partial \ln p}{\partial z}\frac{dz}{dt} = -\frac{g}{RT}\frac{dz}{dt}$$

where $\frac{dz}{dt}$ is rate of change of height following the descending parcels. Note that we have set $\frac{\partial p}{\partial t} \approx 0$ because, in the absence of contrary information, we assume that local rates of change of pressure are negligibly small (this is not quite correct because the calculated thickness changes alone will produce some small local pressure changes, but this effect can readily be shown to be second order). Substituting back into the thermodynamic equation,

$$\frac{\partial T}{\partial t} + \frac{dz}{dt}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right) = \frac{\dot{q}}{c_p} = 0$$

since this is an adiabatic process. Integrating over a day, as before, we get

$$\Delta T = \left[\left(\frac{dT}{dz}\right)_{adiabatic} - \left(\frac{dT}{dz}\right)_{profile}\right]\Delta z.$$

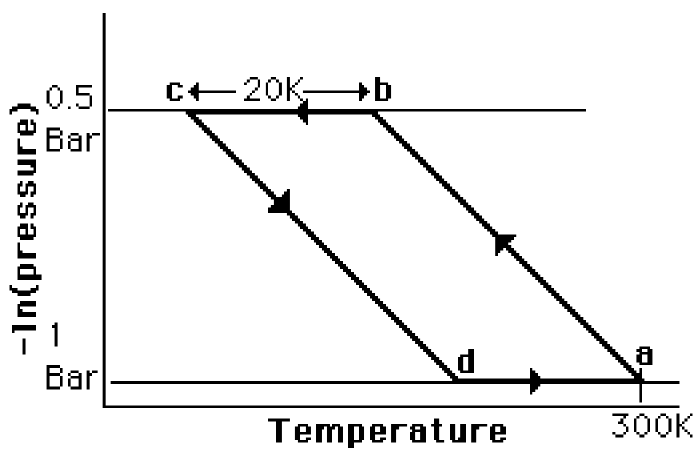
Example 3: The same descending layer loses heat by emission of infrared radiation, mainly to space. What rate of radiative heat loss (in Wm^{-2}) is required to balance the compressional heating due to descent of the layer?

Solution: Use the equation derived above, but with $\frac{\partial T}{\partial t} = 0$ and $\dot{q} \neq 0$.

$$\dot{Q} = \int_{layer} \dot{q}\frac{dp}{g} \approx c_p\left[\frac{dz}{dt}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)\right]\frac{\Delta p}{g} = -1004 Jkg^{-1}K^{-1} x \frac{2.8K}{86400s} x \frac{4x10^4 Pa}{9.81ms^{-2}} = -133Wm^{-2}.$$

Example 4: An air parcel at an initial temperature of 300K and pressure 1000hPa ascends along an isentropic surface to the 500hPa surface. It remains on that pressure surface and slowly cools by emission of radiation to space by 20K. It then descends following an isentropic surface to the 1000hPa surface. There it slowly warms back to its initial temperature as a result of heat added from the underlying surface. Calculate Δh, Δu, Δw, Δq, and $\int \frac{dq}{T}$ for each leg and for the circuit.

Solution: The parcel follows a closed cycle as depicted below with adiabatic trajectories ab and cd and isobaric trajectories bc and da.



Solution: First calculate temperatures at points b and d using Poisson's equation or (equivalently) the definition of potential temperature which is conserved along paths ab and cd: T_b = 246K, T_c = 226K, T_d = 275K. Enthalpy and internal energy changes are:

$\Delta h_{ab} = \int_a^b dh = c_p(T_b - T_a) = 1004(Jkg^{-1}K^{-1})x(-54K) = -54200Jkg^{-1}$,
$\Delta u_{ab} = c_v(T_a - T_b) = -38700Jkg^{-1}$. Similarly $\Delta h_{bc} = -20100Jkg^{-1}$, $\Delta h_{cd} = 49200Jkg^{-1}$, $\Delta h_{da} = 25100Jkg^{-1}$, $\Delta u_{bc} = -14300Jkg^{-1}$, $\Delta u_{cd} = 35100Jkg^{-1}$, $\Delta u_{da} = 17900Jkg^{-1}$.

Work done by parcel expansion against its environment along paths ab and cd can be computed the hard way:

$$\begin{aligned}\Delta w &= \int_a^b pdv = \int_a^b d(pv) - \int_a^b vdp = \int_a^b RdT - \int_a^b RTd\ln p \\ &= R\Delta T_{ab} - R\int_a^b \theta\left(\frac{p}{p_{ref}}\right)^{\kappa}\frac{dp}{p} = R\Delta T_{ab} - \frac{R\theta}{\kappa}\left[\left(\frac{p_b}{p_a}\right)^{\kappa} - 1\right] \\ &= -c_v\Delta T_{ab}\end{aligned}$$

or we can calculate the work done the easy way: $\Delta w = -\Delta u + \Delta q = -\Delta u$ since no heat is exchanged along ab or cd, so that $\Delta w_{ab} = -c_v\Delta T_{ab} = 38720Jkg^{-1}$ and $\Delta w_{cd} = -35130Jkg^{-1}$. Along bc,
$\Delta w_{bc} = \int_b^c pdv = p(v_c - v_b) = R\Delta T_{cb} = -5740Jkg^{-1}$, and $\Delta w_{da} = R\Delta T_{da} = 7180Jkg^{-1}$.

Because work is not a state variable, the net work around the circuit is non-zero, $+5030Jkg^{-1}$. Added heat is zero along paths ab and cd. Along bc, it is
$\delta q_{bc} = \int_b^c [dh - vdp] = \int_b^c dh = \Delta h_{bc} = -20080Jkg^{-1}$, and along da, it is $\delta q_{da} = +25110\,Jkg^{-1}$.
The total added heat around the circuit is $+5030Jkg^{-1}$. Of course, this is equal to the work done around the circuit because $\oint du = \oint[\delta q - \delta w] = 0$.

This example illustrates a central aspect of the atmospheric heat engine. Cycles in which warm air rises and cold air sinks are cycles in which work of expansion is done by air parcels. This work generates the kinetic energy of atmospheric motions, so the prevalence of this "direct circulation" sense of thermodynamic cycles is necessary to maintain the kinetic energy of the winds against dissipation. Sea breeze and Hadley circulations, operating on vastly different scales are examples of such kinetic energy generating circulations. So are deep convective clouds, hurricanes, and, a little less obviously, many (but not all) mid-latitude large-scale weather systems. The simple system analyzed above also shares the general property of heat engines that work on their surroundings: the system gains heat at high temperatures and loses heat at cooler temperatures. For the troposphere, this is accomplished by heat addition by convection from the surface and by release of latent heat of condensation in the lower troposphere and tropics while heat is lost by emission of infrared radiation to space from the upper troposphere and polar regions. Interestingly, the lower stratosphere does not function this way. Heat is added in the cool tropics, and lost from the warmer high latitude stratosphere. The lower stratosphere is driven by the troposphere and acts as a heat engine in reverse.

The last part of this problem, calculation of $\oint \frac{\delta q}{T}$ along each leg is left as an exercise.