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Appendix A

Scientific Notation, Units, and Constants

This appendix reviews the "scientific notation" used in the text; the units commonly used in physics and chemistry to quantify length, mass, and so on; and the manipulation of these units. We discuss some of the physical constants and the essential mathematical operations that are used in the text.

A.1 Scientific Notation

Most scientific discussions use numbers that are either very large or very small. Examples of such numbers can be found in every chapter of this book, including the total number of molecules in the Earth's atmosphere, 10^{44}; the fraction corresponding to a mixing ratio of one part per trillion by volume (pptv), 10^{-12}; the amount of carbon emitted into the atmosphere each year from fossil-fuel combustion, about 10^{19} grams; and the wavelength measure for ultraviolet radiation, roughly 10^{-7} meters. If we had to write out the first number, it would read: 100,000,000,000,000,000,000,000,000,000,000,000,000,000,000. That is, the digit 1 followed by a string of 44 zeros! The second number would read: 0.000000000001. These longhand numbers are clumsy and impractical.

Basically, any large or small number can be expressed as the product of two terms. The first term is a **prefactor** of order unity (that is, a number with a value between 1 and 10) that gives the precision, or accuracy, of the original number. The second term is a **power of 10** (that is, 10^p, where p is the **exponent** or "power" of 10). The power of 10 defines how many times the number 10 is to be multiplied by itself. For example, $10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$ (that is, 10 multiplied by itself four times). As an example of the use of this notation, consider the large number 1,100,000 (which is named one million one hundred thousand). In scientific notation this number may be written 1.1×10^6.

Notice that to arrive at this form, we must count the digits from the position where the decimal point is originally found (or would be placed) in the number toward the left until only one non-zero digit remains to the left. The decimal point is fixed here, and the resulting figure is the prefactor. The number of digits counted to the left becomes the power, or exponent, of 10. The exponent is positive if we have counted from right to left, but is *negative* if we have counted from left to right. In either case, after moving the decimal point, the remaining figure (with one digit remaining to the left of the decimal point) is the prefactor (usually with the unnecessary zeros lopped off). Try this procedure with the number 0.0000000035. In this case, we count from the original decimal point to the right nine digits, leaving the prefactor 3.5 (all the zeros to the left are dropped) and thus yielding the number in the form 3.5×10^{-9}.

The usefulness of scientific notation is easily appreciated for arithmetic with large and small numbers. Imagine that you want to calculate the total number of molecules in the Earth's atmosphere for a species that has a mixing ratio of 1 pptv. This would be written literally as 100,000,000,000,000,000,000,000,000,000,000,000,000,000,000 × 0.000000000001 = 100,000,000,000,000,000,000,000,000,000,000. How much easier is it to write $1 \times 10^{44} \times 1 \times 10^{-12} = 1 \times 10^{32}$. All we do is multiply the prefactors and add the exponents! Consider the examples given later, and enjoy.

Applications of Scientific Notation

To illustrate the use of scientific notation, we offer some examples. The number 550, for example, can be written in scientific notation as 5.5×10^2. That is, $550 = 5.5 \times 10^2$. In this case, the normal decimal representation is more convenient than the scientific representation. Naturally, the simplest notation is adopted for specific numbers in the text of the book. For some numbers, the choice can be a matter of taste. Take, for example, the number of people on the Earth—approximately 5 billion. This may be written out as 5,000,000,000 or more compactly as 5×10^9. You should become familiar with both representations.

If we consider the number of molecules in Earth's atmosphere, there is little dissension. Which number is more convenient to write and which value is more immediately clear: 1×10^{44} or 100,000,000,000, 000,000,000,000,000,000,000,000,000,000,000? In addition to compactness and immediate recognition of magnitude, numbers expressed in scientific notation are much easier to manipulate in arithmetic operations. Some examples of these operations follow. *Multiplication*: The rule for multiplying numbers in scientific notation is to *multiply the prefactors* and *add the exponents* (or the powers of 10). For example,

$$\begin{aligned} 1 \times 10^5 \times 1 \times 10^8 &= (1 \times 1) \times 10^{8+5} \\ &= 1 \times 10^{13} \end{aligned}$$

$$\begin{aligned} 1 \times 10^{-3} \times 1 \times 10^6 &= (1 \times 1) \times 10^{-3+6} \\ &= 1 \times 10^3 \end{aligned}$$

$$\begin{aligned} 1 \times 10^{-2} \times 1 \times 10^{-4} &= (1 \times 1) \times 10^{-2-4} \\ &= 1 \times 10^{-6} \end{aligned}$$

$$\begin{aligned} &(2,500,000,000) \times (200) \\ &\quad = (2.5 \times 10^9) \times (2 \times 10^2) \\ &\quad = (2.5 \times 2) \times (10^9 \times 10^2) \\ &\quad = 5 \times 10^{9+2} \\ &\quad = 5 \times 10^{11} \end{aligned}$$

Division: The rule for dividing numbers in scientific notation is to *divide the prefactors*—that is, to divide the numerator prefactor by denominator prefactor—and *subtract the exponent* of the number in the denominator from the exponent of the number in the numerator. For example,

$$\frac{1 \times 10^4}{1 \times 10^3} = \frac{1}{1} \times 10^{4-3} = 1 \times 10^1 \quad \left(\text{or} \quad \frac{10,000}{1,000} = 10\right)$$

$$\frac{1 \times 10^4}{1 \times 10^5} = \frac{1}{1} \times 10^{4-5} = 1 \times 10^{-1} \quad \left(\text{or} \quad \frac{10,000}{100,000} = 0.1\right)$$

$$\begin{aligned} \frac{1 \times 10^{-6}}{1 \times 10^{-5}} &= \frac{1}{1} \times 10^{(-6)-(-5)} \\ &= 1 \times 10^{-6+5} = 1 \times 10^{-1} \end{aligned}$$

$$\frac{6 \times 10^4}{2 \times 10^3} = \frac{6}{2} \times 10^{4-3} = 3 \times 10^1 \quad \left(\text{or} \quad \frac{60,000}{2,000} = 30\right)$$

$$\frac{5 \times 10^{-3}}{2 \times 10^5} = \frac{5}{2} \times 10^{-3-5} = 2.5 \times 10^{-8}$$

Large and Small Numbers

Tables A.1 and A.2 summarize the notation and identifying names of the most common large and small numbers used in scientific discussions. These numbers may be applied to any specific quantity by attaching an appropriate suffix indicating the units of measure involved.

Using Mixing Ratios

Table A.3 lists the common mixing ratios, or mixing fractions, that are used to characterize gases in the atmosphere. The principal ratios, or fractions, used are parts per million, parts per billion, and parts per trillion (abbreviated ppm, ppb, and ppt, respectively). These mixing fractions can be expressed in parts *by volume (or number)* or parts *by mass* of air. In the former case, for example, one might specify parts per million by volume, or ppmv. For smaller fractions it would be ppbv and pptv. For mixing fractions specified as parts by mass, the corresponding abbreviations are ppmm (for parts per million by mass), ppbm, and pptm, respectively.

"Parts by volume" refers to the *fraction of the total number* of molecules in a unit volume of air of a certain kind of gas—say O_2 or CO_2 or H_2O. The *ratio* of the number of such molecules to the total number of molecules in the volume is the fraction of interest; hence the term *mixing ratio*. An

Table A.1 Large Numbers

"Normal" notation	*Scientific notation*	*Standard name*
1	1×10^{0}	One
10	1×10^{1}	Ten (*deca-*)
100	1×10^{2}	Hundred (*hecto-*)
1000	1×10^{3}	Thousand (*kilo-*)
10,000	1×10^{4}	
100,000	1×10^{5}	
1,000,000	1×10^{6}	Million (*mega-*)
10,000,000	1×10^{7}	
100,000,000	1×10^{8}	
1,000,000,000	1×10^{9}	Billion (*giga-*)
1,000,000,000,000	1×10^{12}	Trillion (*tera-*)

Table A.2 Small Numbers

"Normal" notation	*Scientific notation*	*Standard name*
0.1	1×10^{-1}	Tenth (*deci-*)
0.01	1×10^{-2}	Hundredth (*centi-*)
0.001	1×10^{-3}	Thousandth (*milli-*)
0.0001	1×10^{-4}	
0.00001	1×10^{-5}	
0.000001	1×10^{-6}	Millionth (*micro-*)
0.000000001	1×10^{-9}	Billionth (*nano-*)
0.000000000001	1×10^{-12}	Trillionth (*pico-*)

Table A.3 Mixing Fraction Definitions

One part per million	1 ppm	1×10^{-6}	one out of each million
One part per billion	1 ppb	1×10^{-9}	one out of each billion
One part per trillion	1 ppt	1×10^{-12}	one out of each trillion

easy way to understand the mixing ratio is to imagine grabbing a sample of air in a bottle. You count the total number of individual molecules in the bottle (which would amount to about 2×10^{22} molecules in a quart-size container!). Next, you count the number of specific molecules of interest. If it were carbon dioxide, for example, you would find that the quart bottle held about 7×10^{18} molecules of CO_2. The ratio of these two numbers is $7 \times 10^{18}/2 \times 10^{22} = 3.5 \times 10^{-4}$, which is the same as 350×10^{-6}, or 350 parts per million (by number or volume).

The relationship between volume and number may not be perfectly clear. In fact, the "mixing ratio by volume" and "mixing ratio by number" are equivalent for air. According to the ideal gas law and law of partial pressures (Section 3.1.1), the volume occupied by every molecule in a gas mixture is exactly the same. A molecule of oxygen and a molecule of carbon dioxide take up the same "space" in a container. The molecules themselves are so small that they do not fill a significant part of the volume, and there is no reason to assign more of the remaining space to one molecule or another. Now imagine that all the molecules of a single component of the mixture have been isolated in one section of the container. If they were constrained at the same pressure as that of the original mixture, the fraction of the total volume occupied by that component would be exactly proportional to the fractional number of molecules of that component in the original mixture (we are assuming that the temperature is fixed during these impossible manipulations).

"Parts by mass" is interpreted in a similar way as "parts by volume." For a specified volume of air, the *fraction of the total mass* associated with a certain gas is defined as its mass-mixing ratio in that volume. The volume- and mass-mixing ratios are, of course, related. The mass mixing fraction of a certain component is its volume-mixing fraction multiplied by its molecular weight and divided by the average molecular weight of air. More directly, the mass-mixing ratio is the ratio of the density of the component to the density of air in the parcel of interest. The following algebraic relations hold:

$$
\begin{aligned}
r_{i,m} &= \frac{\rho_i}{\rho_A} \\
&= \frac{m_i n_i}{m_A n_A} = \frac{m_i}{m_A}\frac{n_i}{n_A} = \frac{m_i}{m_A} r_{i,v}
\end{aligned}
\qquad \text{(A.1)}
$$

where $r_{i,m}$ is the mass-mixing ratio of a gas species i in air, $r_{i,v}$ is its equivalent volume-mixing ratio, and ρ_A and m_A are the density (mass/volume) and average molecular weight (atomic mass units, amu) for air, respectively.

In this book, we use volume or number and mixing ratios or fractions wherever possible. Some books and research papers use mass-mixing fractions. They are easily converted to volume fractions when the molecular weight of the gas is known.

A.2 The Metric System: Units and Conversions

All physical parameters have units by which they are measured. For example, **time**, **length**, and **mass** are basic physical parameters that, unless given specific values, are not particularly useful in quantitative work. Similarly, temperature and electrical charge are basic physical parameters. (See Section 2.1.3 for a further discussion of temperature.) The units for each parameter provide a point of reference that defines the magnitude of the parameter. For example, time may be specified in seconds, hours, days, weeks, months, years, decades, centuries, millennia, and so on in smaller intervals such as milliseconds and microseconds. Accordingly, any specific interval of time can be expressed as a specific number of units of time. As long as the basic unit of time itself is well defined by some standard or reference, any given interval can then be precisely specified in those units. In almost all scientific systems of units, seconds are the basic measure of time. Likewise, kelvin (K) is the preferred unit for temperature.

A system of units is the collection of basic units for all the physical parameters of interest. For example, the **mks metric system** of units consists of lengths given in meters, mass in kilograms, and time in seconds. A related system of metric measure called the **cgs** system specifies lengths in centimeters, mass in grams, and time in seconds. Throughout this book, an attempt is made to conform to the *Système Internationale d'unités* (SI) designation for all units. The SI units are essentially the mks metric system units. The Congress of the United States legalized the use of the metric system in 1866. This acceptance was reinforced by the United States' signing of the Treaty of the Meter in 1875. Before that, only measurements in the British system (feet, pounds, quarts) were legally accepted. Even so, to this day Americans remain addicted to the more difficult and scientifically abandoned British system of units.

Other physical parameters deriving from the basic parameters are given units consistent with the basic units. For example, volume has dimensions of length times height times width. All three dimensions are actually length, so volume has overall units corresponding to length cubed, or l^3 (later we explain "cubed"). Thus volume has units of meters cubed (m^3) in the mks metric system and centimeters cubed (cm^3) in the cgs metric system. There is an important distinction between the terms *dimensions* and *units* as they are used here. Dimensions define the way that a quantity depends on basic physical parameters, and units define the scale by which the dimensions are measured. Thus length (l) is a basic dimension, and meter is a unit of length. Similarly, mass (m) is a basic dimension, or physical attribute, and kilogram is a specific measure of the amount of mass. Time (t) is also a basic dimension, and seconds are one of the possible corresponding units.

Force depends on the basic parameters as mass times length divided by time squared, or $m \times l/t^2$. Note that the dimension of mass as indicated by the abbreviation, m, in the dimensional expression, should not be confused with meter, which is a unit of length, l. Likewise, the dimension of time, which is abbreviated as t in dimensional expressions, should not be confused with the metric ton, or tonne, a unit of mass (also abbreviated t). One must be careful to distinguish between a dimension and its unit of measure, and between a dimensional expression and its equivalent combination of units. For example, in the mks metric system, the combination of units corresponding to the dimensional expression for force, $m \times l/t^2$, are kilogram-meter/second-second, or $kg{\cdot}m/sec^2$. The newton (N) is defined as 1 $kg{\cdot}m/sec^2$.

Energy can be defined as force times distance, which depends on the basic physical dimensions as $m \times l^2/t^2$, with corresponding metric units of kilogram-meter-meter/second-second, or $kg{\cdot}m^2/sec^2$. The joule (J) is defined as a 1 newton-meter (N-m), or 1 $kg{\cdot}m^2/sec^2$.

More complex parameters, such as a radiant energy flux, can also be defined in terms of the basic dimensions. For example, any energy flux may be defined as energy per area per time, which can be decomposed into a dependence on basic parameter dimensions as, $(m \times l^2/t^2)/(l^2 \times t) = m/t^3$. We have several equivalent combinations of metric units for the energy flux: $J/m^2{\cdot}sec$ = kilograms/sec^3 = watts/m^2 (where only the second form is given in basic units). The watt (W) is the specific unit of power in the metric system, where 1 watt is 1 J/sec.

Common Units of Measure

We next compare the most frequently employed units of measurement for length and mass in different systems of units. Note that one of the basic physical parameters (and units) that is not treated explicitly in this appendix is **electrical charge**. Electrical charge is important to the development of electromagnetic theory, which has not been discussed in the text. Accordingly, the unit of charge and the units deriving from it, are not defined further.

Length

1 meter (m) = 100 centimeters (cm)= 1000 millimeters (mm)
1000 m = 1 kilometer (km)
1 m = 1×10^6 micrometers (mm); 1 mm = 1×10^{-6} m
1 m ≅ 39 inches (the symbol ≅ means "very closely, but not exactly, equals")
100 m ≅ 110 yards
1 inch ≅ 2.5 cm
1 mile ≅ 1.6 km

Volume

1 liter (l) = 0.001 cubic meter (m^3) = 1000 cubic centimeters (cm^3)
1 liter ≅ 61 cubic inches ≅ 1.06 quarts

Mass

1 kilogram (kg) = 1000 grams (g)
1 metric ton (t) = 1 tonne = 1000 kg = 1×10^6 grams (g)
1 kg ≅ 2.2 pounds
450 g ≅ 1 pound
1 tonne ≅ 2200 pounds = 1.1 English ton (or ton)

Among the many derivative units that can be defined, such as the newton, joule, and watt, are the following useful units. Note that pressure is defined as force per unit area.

Pressure

1 pascal (Pa) = 1 newton/meter^2 (1 N/m^2; or 1 kg/m·sec^2)
100,000 Pa = 1 bar (roughly the average pressure at the Earth's surface)
1000 millibar (mb) = 1 bar
1 mb = 100 Pa
1 atmosphere (atm) ≅ 1013 mb
1 atm ≅ 1 bar ≅ 14.7 pounds/inch^2

Manipulation of Dimensions and Units

Dimensions of parameters and their corresponding units can be treated like variables in an equation. The symbols representing the dimensions of length (l), mass (m), and time (t) can be manipulated in any expression in which the original physical quantities are multiplied or divided. We did this when manipulating the basic dimensions of derived quantities such as force and energy flux. We can write out similar relationships for the units corresponding to the dimensions. Thus, $m \times l/t^2$ is equivalent to kg·m/sec^2. In more complex combinations of dimensions or units, we can cancel dimensions or units that appear in both the numerator and denominator and collect dimensions or units that are multiplied.

One of the most important rules regarding the use of units is that whenever they are combined in any expression, *the units must always be consistent.* One should never, for example, mix together mks and cgs units; or use seconds in one part of an expression and years in another, unless it is clearly established that such usage will not cause problems. Accordingly, if an expression with two or more parameters is being evaluated, you have two choices: (1) Make sure all the parameters are in the same units (mks, say) or (2) correct the units of the individual parameters, or the final expression, by multiplying by appropriate **conversion factors**.

The principal guidelines for the conversion of units can be demonstrated using a simple example. Suppose you want to evaluate the distance that an air mass had traveled in a certain time and you know the velocity (speed) of the parcel. You will need to evaluate the expression

$$d = v\tau \qquad \text{(A.2)}$$

where d is the distance, v is the velocity, and τ is the time. The velocity has units of length per unit time, or l/t. The basic dimensions are consistent because Equation A.2 can be written in dimensional form as, $l = (l/t) \times t = l$. Notice that the dimension of time cancels out on the right-hand side of Equation A.2.

If velocity is measured in meters per second and the time is measured in seconds, then clearly the distance will be in meters. This is a consistent set of units. However, if the velocity is given in m/sec and the time is given in hours, the units will be inconsistent. We must convert the parameters to consistent values, by using a conversion factor. In this case, we note that 1 hour = 3600 seconds and so write Equation A.2 as follows:

$$d\,(\text{meters}) = v\left(\frac{\text{meters}}{\text{second}}\right) \times \tau\,(\text{hours}) \times \left(\frac{3600 \text{ seconds}}{1 \text{ hour}}\right) = 3600\, v\tau \qquad \text{(A.3)}$$

The converted equation can now be evaluated using the original unconverted values of the velocity and time parameters. The conversion factor that has been applied in this example is the last term in the middle section of Equation A.3. Notice that this term changes the units of hours in the numerator to units of seconds by multiplying by the number of seconds per hour.

Another example follows from the box model equation for the concentration, q, of a material in a reservoir (see Equation 4.1, for example):

$$q = \frac{S\tau}{V} \qquad \text{(A.4)}$$

where S is the source, τ the residence time, and V the volume of the reservoir. Imagine that the source is given in tonnes per day, the residence time is given in hours, the volume is specified in cubic kilometers, and the concentration is in grams per cubic centimeter. Clearly, each of the parameters determining q needs to be converted to the appropriate units.

Instead of illustrating this procedure step by step, the various conversions are shown collectively in a modified version of Equation A.4:

$$q\left(\frac{\text{g}}{\text{cm}^3}\right) = \frac{S\left(\frac{\text{tonne}}{\text{day}}\right)\left(1 \times 10^6 \,\frac{\text{g}}{\text{tonne}}\right)\tau\,(\text{hours})\left(\frac{1 \text{ day}}{24 \text{ hours}}\right)}{V\left(\text{km}^3\right)\left[\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{100 \text{ cm}}{1 \text{ m}}\right)\right]^3}$$

$$= \frac{S\tau}{V}\left[\frac{\frac{1\times10^{6}}{24}}{(1000\times100)^{3}}\right] = \frac{S\tau}{V}\left(\frac{1\times10^{-9}}{24}\right) \quad (A.5)$$

In this massive equation, the same simple form of the original equation is retained. Each of the parameters on the right-hand side of the equation is operated on by a specific conversion factor. The conversion factors simply relate the original units to the desired units, based on the relationships between the common units of measure listed earlier. Some of the conversions require more than one step. For example, to get from kilometers to centimeters, we first convert kilometers to meters, and then convert meters to centimeters. This could be done in one step if the relationship between kilometers and centimeters were already known—that is, 1 km = 1×10^5 cm. The final overall conversion factor in Equation A.5 combines all the individual conversion steps by straightforward multiplication and division.

A.3 Physical and Mathematical Constants

In physics, chemistry, and mathematics, certain numbers are special. These special numbers are usually constants that make general relationships between parameters into exact relationships with the appropriate units. For example, the thermal energy, E, of a molecule in a gas is proportional to the temperature, T, of the gas. But the energy is given quantitatively by the relation

$$E = \frac{3}{2} k_B T \quad (A.6)$$

Here the physical "constant," k_B, fixes the amount of energy. It was originally determined by the physicist Ludwig Boltzmann, and so is called **Boltzmann's constant**. Many simple and useful relations in science contain a fundamental constant such as Boltzmann's constant. Some of these, which are referred to in the chapters of this book, are listed next.

Physical Constants (and Their Common Symbols)

The following constants have associated units of measure for the mks metric system of units.

Universal Constants of Physics and Chemistry

Atomic unit mass, $m_0 = 1.67 \times 10^{-27}$ kg/amu: The atomic unit mass is equivalent to the mass of a proton or neutron in an atomic nucleus. If the total number of protons and neutrons in a nucleus is known—that is, the number of atomic mass units, or "amu"—the total mass of the nucleus will be the number of amu multiplied by the unit mass, m_0. For convenience, the weights of atoms and molecules are stated in amu rather than kilograms. The weight of 1 mole, or Avogadro's number, of atoms or molecules is equal in grams to the number of amu; that is, $m_0 N_A = 1 \times 10^{-3}$ kg/amu = 1 g/amu.

Avogadro's number, $N_A = 6.02 \times 10^{23}$ molecules/mole: Avogadro's constant defines a "mole" of a substance; it is exactly N_A molecules—or smallest molecular entities—of that substance. The mass of a mole of a substance is equal to the atomic weight of the substance expressed in grams. For example, the atomic weight of ozone (O_3) is 48, equivalent to the combined atomic weight of three oxygen atoms. One mole of ozone therefore is the same as 48 grams. In other words, 6.02×10^{23} ozone molecules weighs 48 g.

Boltzmann's constant, $k_B = 1.38 \times 10^{-23}$ J/K: Boltzmann's constant relates temperature to the thermal energy of motion of single molecules. To emphasize this point, the units of k_B can be written as J/K-molecule. Boltzmann's constant is the most frequently used physical constant in descriptions of gases and their mechanical and thermodynamic behavior.

Gas constant, $R_g = 8.31$ J/K-mole: The "universal" gas constant is closely related to Boltzmann's constant, since both connect temperature with energy. In this case, the energy is per *mole* of gas, rather than energy per molecule. As you might suspect, the gas constant can be defined in terms of Boltzmann's constant and Avogadro's number as $R_g = k_B N_A$. This equivalence is easy to check.

Gravitational constant, $G = 6.67 \times 10^{-11}$ N·m^2/kg^2: The gravitational constant quantifies the gravitational force that one body of a given mass exerts on another body of known mass. The law of gravitation applies to *all* objects regardless of size. The smallest objects exert some gravitational force on the largest, although only the gravity of very large objects, such as the Earth, is obvious.

Planck's constant, $h = 6.63 \times 10^{-34}$ J·sec: Planck's constant relates the energy of a photon of radiation to the frequency of the radiation. (Frequency and wavelength are connected through Equation 3.18.) Only the frequency or wavelength of light needs to be known in order to determine the energy of a photon of radiation at that frequency or wavelength (assuming that the radiation is traveling in free space or in air where the speed of light is constant).

Speed of light, $c = 3.00 \times 10^8$ m/sec: The speed, or velocity, at which light, or *any other radiation*, moves in a vacuum is exactly determined as 3×10^8 m/sec. Moreover, the velocity of a photon is independent of wavelength (or frequency) and holds for all photons. The speed of light in other materials (air, glass, water, and so forth) is slower, although the velocity in air is almost exactly the same as in a vacuum.

Stefan-Boltzmann constant, $\sigma_B = 5.67 \times 10^{-8}$ W/m^2·K^4: The Stefan-Boltzmann constant relates the power emitted by a blackbody, per unit surface area of the blackbody, to the fourth power of its temperature (Section 3.2.1).

Wien constant = 2900 μm·K: The Wien constant can be derived from Planck's constant for blackbody radiation and relates the temperature of the blackbody to the wavelength at which the radiant energy is the most intense (Section 3.2.1).

"Constants" of the Physical World

The following physical "constants" have been used in the text of this book or may be of general use to those studying the atmospheric environment. Note that in several cases, the "constants" are not truly constant, but may vary by some small amount. In many applications, particularly at the level of detail required in this book, they may nevertheless be treated as true constants.

Angular speed of rotation of Earth, $\Omega = 7.29 \times 10^{-5}$ radians/sec: The Earth rotates on its axis at a nearly constant "angular velocity." The unit of angular motion is the "radian"; there are exactly 2π radians in a single rotation about an axis (later we define π). One radian equals about 57 degrees of angle. The Earth rotates through 2π radians, or 360 degrees of angle, in somewhat less than one day, or 86,400 seconds.

Atmospheric pressure at sea level = 1013 mb = 1.013×10^5 N/m^2: This is the average pressure of the atmosphere.

Density of air at sea level = 1.225 kg/m^3: The average density of the atmosphere at sea level is determined by the average surface pressure and temperature.

Gas constant for dry air (R_g^*) = 287 J/K·kg: The gas constant for dry air can be obtained by dividing the "universal" gas constant by the average molecular weight of air in amu, or about 29 g/mole, and multiplying by 1000 to convert from grams to kilograms. The gas constant says that each kilogram of air holds 287 joules of thermal energy for each degree kelvin of its temperature.

Gravitational acceleration (g) = 9.81 m/sec^2: The gravitational *force* exerted by the Earth on an object at sea level is simply the *weight* of the object. The weight is the mass of the object multiplied by the gravitational acceleration, $W = mg$. Often, an object's "weight" and "mass" may be treated as interchangeable parameters; the equivalence derives from the constancy of the gravitational acceleration, which relates them.

Mass of the Earth = 6.00×10^{24} kg: The masses of the major components of the Earth system are discussed in Sections 2.3.1 and 4.1.1.

Radius of the Earth = 6.37×10^6 m: This is the average radius of the Earth. The Earth tends to be somewhat flattened at the poles rather than perfectly spherical; the greatest radius is at the equator because rotation on an axis creates a centripetal force that causes the equatorial regions to bulge.

Solar constant (F_S) = 1390 W/m^2: The solar constant is discussed in Section 11.3.1.

Sun-Earth distance = 1.50×10^8 m: The mean distance of the Earth from the sun corresponds to the orbit of the Earth around the sun. The difference between the maximum and minimum distances is about 5 percent of the average distance. Accordingly, the distance between the sun and the Earth is only roughly constant.

Mathematical Constants

π: A mathematical constant that is exactly equal to the ratio of the circumference of a circle to its diameter, π also gives the area of a circle when multiplied by the radius of the circle squared. (See Section A.4 for a definition of "squared.") There are exactly 2π angular radians in one rotation or revolution about a point or axis: π = 3.1416.

e: The base of the **natural logarithms**, or *ln*, is $e^{\ln(x)} = x$ (Section A4). Also, e is the base for the exponentiation in the exponential function, exp; e = 2.718.

A.4 Mathematical Operations

The principal mathematical operations used in this text are taking the "square," and its complement, taking the "square root," using the natural exponential operation, and algebraically manipulating simple equations. These operations are summarized next.

Squares and Square Roots

The "square" of a number is simply the number multiplied by itself, so the square of 2 is $2 \times 2 = 4$. The square of 3 is $3 \times 3 = 9$, and so on. The square of any number can be written in the form

$$s(n) = n \times n = n^2 \quad \text{(A.7)}$$

where $s(n)$ is the "square of n," and n is the number to be "squared." The exponent shown as a superscript of n is the "power" to which n is raised—here, the second power. The second power of n is the same as the square of n. The number n may be a very large number, a fraction, or any physical or mathematical constant.

The operation that is opposite to squaring a number is taking its "square root." The special mathematical symbol, $\sqrt{}$, is used to indicate a square root. The square-root operation determines the quantity that, if multiplied by itself, will yield the original number. For example, we know that $2 \times 2 = 4$; therefore, 2 must be the square-root of 4, or $\sqrt{4} = 2$. There is nothing mysterious here, just a little arithmetic. The square root of a number can be written in shorthand notation as

$$r(n) = \sqrt{n}\,;\; r \times r = n = n \times n = n^2 \quad \text{(A.8)}$$

It follows that n is the square of r. The square root may also be written as a the fractional one-half power of a number, such as: $r = \sqrt{n} = n^{1/2}$. Further relationships may then be written, for example:

$$\sqrt{s(n)} = \sqrt{n^2} = (n^2)^{1/2} = n^{2 \times 1/2} = n^1 = n \quad \text{(A.9)}$$

This relation shows that the square root of the square of n is n. A key point is that when a number is raised to a "power" in two or more consecutive operations, the final effect can be obtained by multiplying together the separate powers into one number. As you might expect, if you undo a mathematical operation you just performed with an exactly complementary operation, you will end up at the starting point.

Higher Powers

We are not restricted to "squares" in multiplying a number by itself. It would be just as easy to multiply a number by itself three times; or four times, and so forth. The cube of a number is that number raised to the "third power," or multiplied by itself three times. Thus the cube, or third power, of 2 is $2^3 = 2 \times 2 \times 2 = 8$. The cube of 3 is $3^3 = 3 \times 3 \times 3 = 27$. And so on. The "cube root" is also defined as for the square root: The cube root is the one-third power of a number that, when multiplied by itself three times, returns the original number. Hence the cube root of 8 is $8^{1/3} = (2 \times 2 \times 2)^{1/3}\ (2^3)^{1/3} = 2$. One only need perform a complementary operation to obtain a root (although this operation is not always so easy).

These arguments may be extended to the "fourth power." The fourth power of 2 is $2^4 = 2 \times 2 \times 2 \times 2 = 16$; for 3, it is $34 = 3 \times 3 \times 3 \times 3 = 81$. The "fourth root" is defined in a similar way as for the other roots. The fourth root is the one-fourth power of a quantity. In the case of the number 16, the fourth root is 2, found as $16^{1/4} = (2^4)^{1/4} = 2$. One trick (you will perhaps not find so entertaining) is to find the fourth root of a quantity by taking the "square root" twice, or the square root of the square root. This is illustrated for the number 16:

$$16^{1/4} = (16^{1/2})^{1/2} = \sqrt{\sqrt{16}} = \sqrt{4} = 2 \quad \text{(A.10)}$$

Notice that the square root of 16 is 4 (that is, $4 \times 4 = 16$), and the square root of 4 is 2. The one-fourth power is equivalent to applying two one-half powers consecutively, since $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$.

Exponentials and Logarithms

Exponential functions are commonly used in physics and mathematics. Many fundamental processes can be described by "exponential" behavior. The exponential of a number, n, is defined by the mathematical expression

$$\exp(n) = e^n;\ \exp(-n) = e^{-n} = \frac{1}{e^n} \quad \text{(A.11)}$$

The number, n, is the "exponent" in this relation. The mathematical constant, e, is called the "base of the natural logarithms" (its value was given earlier). Notice that when an exponential function is inverted, only the sign of the exponent changes (the last two relations in Equation A.11). Any number, n, can be placed in an exponential function. Moreover, the exponential function is *always positive*, even when the exponent is negative. The exponential of "zero" is exactly "one": $\exp(0) = 1$.

The logarithmic function is the complement of the exponential function. Taking the logarithm of an exponential yields the exponent. The logarithmic function is represented by the symbol *ln*.[1] Then the following relations hold:

$$\ln[\exp(n)] = \ln e^n = n; \ \ln e = 1 \qquad \text{(A.12)}$$

We can calculate (or "take") the logarithm of any *positive* number. Although the logarithm of a negative number is not defined, the value of a logarithm may be negative if the original number is smaller than 1. The logarithm of "one" is exactly "zero": $\ln 1 = 0$.

If x is a number larger than zero, then its logarithm is $m = \ln x$. Because the exponential and logarithmic operations are complementary, it follows that

$$\exp(\ln x) = e^{\ln x} = e^m = x \qquad \text{(A.13)}$$

Accordingly, any positive number can be written as an exponential of a base number, which is the base of the logarithms (Footnote 1).

The exponential and logarithmic functions are related to the scientific notation discussed earlier in this appendix. It turns out that $\ln 10 = 2.3$. Then the following relations hold:

$$\exp(2.3n) = e^{2.3n} = (e^{2.3})^n = 10^n \qquad \text{(A.14)}$$

In other words, taking an exponential is similar to dealing with powers of 10 in scientific notation. When multiplying or dividing numbers, exponents may be added or subtracted, respectively, as in scientific notation. For example:

1. Logarithms that are calculated using the mathematical constant e as the base are called *natural* logarithms. This is sometimes written as $\ln_e$, although the symbol ln represents the natural logarithm. Logarithms can be taken with respect to any base as long as the base has a positive value. The most common alternative base for logarithms is 10 (that is, $\ln_{10}$), which is consistent with the representation of large and small numbers in powers of 10, or in scientific notation.

$$\begin{aligned} \exp(n) \times \exp(m) &= \exp(n + m); \\ e^n e^m &= e^{n+m}; \\ 10^{2.3n} \times 10^{2.3m} &= 10^{2.3(n+m)} \end{aligned} \qquad \text{(A.15)}$$

The division of exponential functions, like the division of numbers expressed in powers of 10, is equivalent to subtracting the exponents:

$$\begin{aligned} \exp(n) \div \exp(m) &= \exp(n - m); \\ \frac{e^n}{e^m} &= e^{n-m}; \\ 10^{2.3n} \div 10^{2.3m} &= 10^{2.3(n-m)} \end{aligned} \qquad \text{(A.16)}$$

Although the terminology and notation used to describe exponentials and logarithms may seem a bit obscure, these functions are indispensable to mathematicians and scientists. They are an important part of the shorthand language of science.

Algebraic Equations

We have been using equals signs throughout this appendix to indicate equivalence between two numbers or quantities or functionalities. Such "equations" are nothing more than a statement that two specific quantities are equal. Trivial equations of the form $x = x$ (for example, $2 = 2$), convey no information; they are obvious identities. But more general equations that connect different parameters or functions can convey useful information in a compact and convenient form. Thus the equation $x = y$ simply states that the quantity x is equal to the quantity y. It should be clear that this statement can be inverted to say y is equal to x. The two statements are perfectly compatible, given the equality. In fact, all the equations in this book can be interpreted in either direction.

If either x or y or both are functions of some sort, the equation may be an "algebraic" equation. For example, we may have the relation

$$x = y^2 \qquad \text{(A.17)}$$

This equation represents a straightforward statement that x equals the square of y, which establishes the unequivocal relationship between x and y. Indeed, from our previous discussion, we can easily deduce that y is the *square root* of x. That is, the equation can be inverted, or solved for y in terms of x. In this case, the procedure is simple: We take the square root on each side of the equals sign and switch sides to get

$$y = \sqrt{x}, \text{ or } y = x^{1/2} \tag{A.18}$$

In any equation, we can apply the same operation to both sides of the equals sign without changing the equality. In this example, we took the square root. We could also have divided each side by the same constant or taken the fourth root of each side, or taken the logarithm, or carried out any number of operations sequentially. Usually, the operations are chosen to simplify one side of the equation or to isolate one of the variables or parameters on one side of the equals sign. In the preceding example, we isolated y on the left-hand side of the equation (which is usually the side chosen for the single variable).

We can take a concrete example from Chapter 11, in which Equation 11.12 was derived from the energy balance model of the Earth:

$$F_S(1 - \alpha_e) = 4\sigma_B T_e^4 \tag{A.19}$$

In this equation, we could have isolated, or solved for, any of the parameters or variables. Typically, one variable is to be determined from all the other information given. In this instance, it is the temperature, T_e. This can be accomplished by performing the following sequence of operations on each side of the equation: Divide by 4; divide by σ_B; take the one-fourth power of each side; reverse the sides of the equation. Having performed these steps, you will have derived Equation 11.14:

$$T_e = \left[\frac{F_S(1 - \alpha_e)}{4\sigma_B}\right]^{\frac{1}{4}} \tag{A.20}$$

Equations are fun to work with because they are concise, precise statements of facts about the world. So do not be intimidated by these wonderful little artifacts of science.

INEQUALITIES

Not all algebraic relations are exact. Equations are exact when they are connected with an "equals" sign (=). But we sometimes wish to express a conditional relationship between two parameters. For example, it may be important to know when one quantity is "less than" another or "greater than" it. Such conditions are expressed in a concise way by means of special mathematical symbols. The symbol $<$ is literally interpreted as "less than." If the condition $x < 0$ holds, then the value of x is always "less than" zero. It can be $-1, -2, -10^6$, or any other negative number. Related symbols are $>$, which means "greater than"; $\leq$, which translates to "less than or equal to"; and $\geq$, which means "greater than or equal to." The inequalities express a certain degree of uncertainty in a relationship and, at the same time, a degree of certainty. Someone may admit to being older than you but may not say by how much. In that case, you could mathematically write

their age > my age

You are using a mathematician's shorthand.