AMATH/ATMOS 505, OCEAN 511—Autumn 03

Homework 3

1. Determine how the vorticity equation for inviscid barotropic flow

$$\frac{D\omega_i}{Dt} = (\vec{\omega} \cdot \nabla) u_i - (\nabla \cdot \vec{u}) \omega_i.$$

simplifies for two dimensional flows. That is, assume that $\vec{x} = (x, y, z)$, $\vec{u} = (u, v, 0)$, $\vec{\omega} = (\xi, \eta, \zeta)$ and that all variables are independent of z.

(a) Derive (trivial) equations for ξ and η, and the equation for $D\zeta/Dt$.

(b) Under what additional conditions is ζ conserved along a Lagrangian trajectory following the flow?

2. Show that for inviscid barotropic flow

$$\frac{D}{Dt}\left(\frac{\omega_i}{\rho}\right) = e_{ij}\left(\frac{\omega_j}{\rho}\right).$$

where e_{ij} is the strain rate tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Hint: As part of the derivation, it will be helpful to decompose $\partial u_i/\partial x_j$ into e_{ij} and an anti-symmetric piece that can be simplified using the relation

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = -\epsilon_{ijk}\omega_k.$$

You may use this last relation without pausing to prove it in your derivation. [It can be proved easily using the identity (2.19) on p. 36 of Kundu and Cohen.]

Interpretation: your result implies that the changes in $\vec{\omega}/\rho$ following the flow are due to stretching and tilting of the vortex lines as the fluid element deforms. (See p. 58 of Kundu and Cohen for a discussion and interpretation of the strain rate tensor.) In fact as proved by Cauchy, the changes in $\vec{\omega}/\rho$ due to flow deformation are proportional to those undergone by an infinitesimal segment of the vortex line $\vec{\delta\ell}$ a the same point in the fluid.

3. Use summation notation to derive the vector identity

$$\nabla \times (\vec{a} \times \vec{b}) = \vec{a}\nabla \cdot \vec{b} + (\vec{b} \cdot \nabla)\vec{a} - \vec{b}\nabla \cdot \vec{a} - (\vec{a} \cdot \nabla)\vec{b},$$

which was used in the derivation of the vorticity equation in class and on p. 136 of Kundu and Cohen.

4. The *nondivergent* velocity field "induced" by (or associated with) an isolated straight vortex line of strength ζ_0 is a cylindrically circular flow around the line with tangential velocity at a distance r perpendicular to the vortex line of $u_T = \zeta_0/(2\pi r)$. Given this, show that the velocity field induced by a planar sheet of straight vortex lines is a piecewise uniform flow with a discontinuous jump at the vortex sheet itself, such that if the vortex sheet lies in the x-y plane $z = 0$, with individual vortex lines oriented parallel to the positive y-axis, then $v = w = 0$ and

$$u = \begin{cases} \zeta_0/2 & \text{if } z > 0; \\ -\zeta_0/2 & \text{if } z < 0. \end{cases}$$

Hint: Both the vorticity distribution and the velocity fields are uniform in y, so this coordinate can be neglected. Compute the velocity at an arbitrary point (x_0, z_0) by integrating the contributions from all the individual line vortices over the range $-\infty < x < \infty$.

Due Friday November 7th