Plane-Wave Summary

by Greg Hakim and Dale Durran

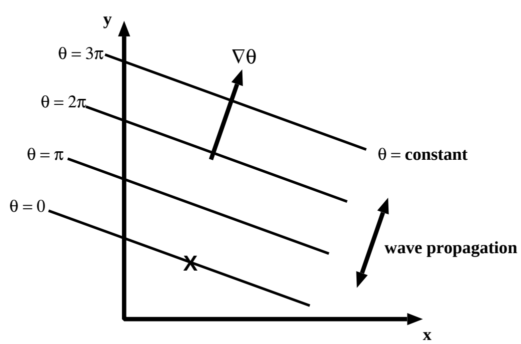
A two-dimensional plane wave may be expressed as

$$f(x,y,t) \;=\; Re\left\{\, A\, e^{i(kx+ly-\nu t)} \right\} \;=\; Re\left\{\, A\, e^{i\theta} \right\} \qquad (1)$$

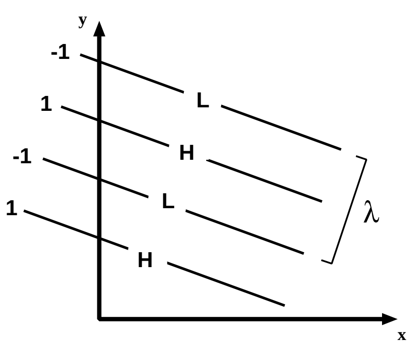
- x, y and t are independent variables (space and time).
- k and l are the x and y *wavenumbers* (units: m^{-1}).
- A is the wave *amplitude*.
- $\theta = kx + ly - \nu t$ is the wave *phase angle*.
- The wave *propagates* normal to lines of constant phase angle.

At any instant in time [t fixed; (x, y) varies]

Plot of θ as a function of (x,y) for fixed t.



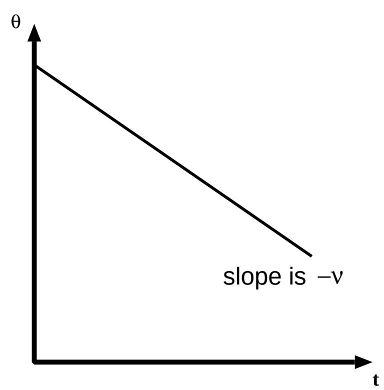
Plot of *Re{exp(i* θ)} as a function of (x,y) for fixed t.



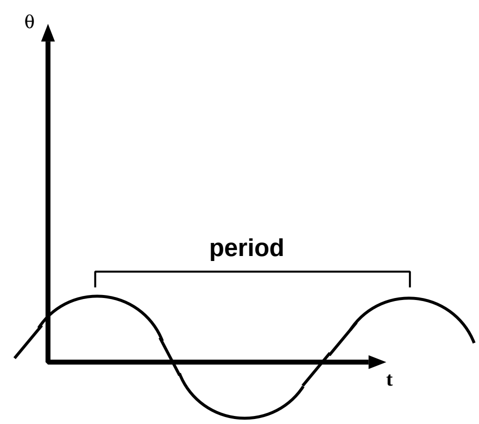
- $\theta = kx + ly + C$; θ is a linear function of space.
- θ is constant on lines of $kx + ly$.
- $e^{i\theta} = e^{i(\theta + 2\pi n)}$, where n is an integer, are lines of *equal phase* (e.g. all highs, lows, nodes, etc).
- $\vec{K} = \nabla\theta = \vec{i}k + \vec{j}l$ is the *wave vector*; $\mathcal{K} = |\vec{K}|$ is the *wavenumber*.
- $\lambda = \frac{2\pi}{\mathcal{K}}$ is the *wavelength*: the distance between neighboring lines of equal phase.

At any fixed point in space $[(x, y)$ fixed; t varies$]$

Plot of θ as a function of t for fixed (x,y).



Plot of *Re{exp(i* θ)} as a function of t for fixed (x,y).



- $\theta = C - \nu t$; θ is a linear function of time.
- $\nu = -\dfrac{\partial\theta}{\partial t}$, is called the *frequency*: the rate that lines of constant phase pass a fixed point in space (units: s^{-1}).
- The wave *period* is $\dfrac{2\pi}{\nu}$: length of time between points of constant phase (units: s).

Stable and unstable waves

If θ has an *imaginary part*, $\theta = \theta_r + i\theta_i$, then $e^{i\theta} = e^{i(\theta_r + i\theta_i)} = e^{i\theta_r}\, e^{-\theta_i} \equiv A^* e^{i\theta_r}$. θ_r is the wave phase angle as interpreted above, and $A^* = Ae^{-\theta_i}$ is a modified amplitude that depends on time and/or space. For example, if the frequency, ν, contributes the imaginary part, then the wave has *time-dependent* amplitude that grows or decays with time. Growing waves are called *unstable*, to distinguish them from the *neutral* waves ($A =$ constant) that we discussed above.

Phase speed and trace speed

- The *phase speed* is the propagation speed of constant phase lines in the direction of $\vec{K}$, $c = \dfrac{\nu}{\mathcal{K}} = -\dfrac{1}{|\nabla\theta|}\dfrac{\partial\theta}{\partial t}$ (units: m s^{-1}).

• The *trace speeds* are the speeds at which lines of constant phase propagate parallel to the coordinate axes. These speeds are *never smaller* than the true phase speed. The x trace speed is ν/k, the y trace speed is ν/l. Note in particular that the apparent phase speeds along each axis (ν/k and ν/l) do not add vectorially to give the true phase speed ($\nu/\mathcal{K}$).

• In principle k, l and ν can all have arbitrary sign, but if this is allowed a redundancy is introduced in the representation of the waves. The following figure shows the four possible wave configurations that can exist for a given set of $|k|$, $|l|$, $|\nu|$. Each wave as the following characteristics:

(a) $\mathrm{sgn}(k/l) < 0, \quad \mathrm{sgn}(\nu/k) < 0, \quad \mathrm{sgn}(\nu/l) > 0$

(b) $\mathrm{sgn}(k/l) > 0, \quad \mathrm{sgn}(\nu/k) > 0, \quad \mathrm{sgn}(\nu/l) > 0$

(c) $\mathrm{sgn}(k/l) > 0, \quad \mathrm{sgn}(\nu/k) < 0, \quad \mathrm{sgn}(\nu/l) < 0$

(d) $\mathrm{sgn}(k/l) < 0, \quad \mathrm{sgn}(\nu/k) > 0, \quad \mathrm{sgn}(\nu/l) < 0$

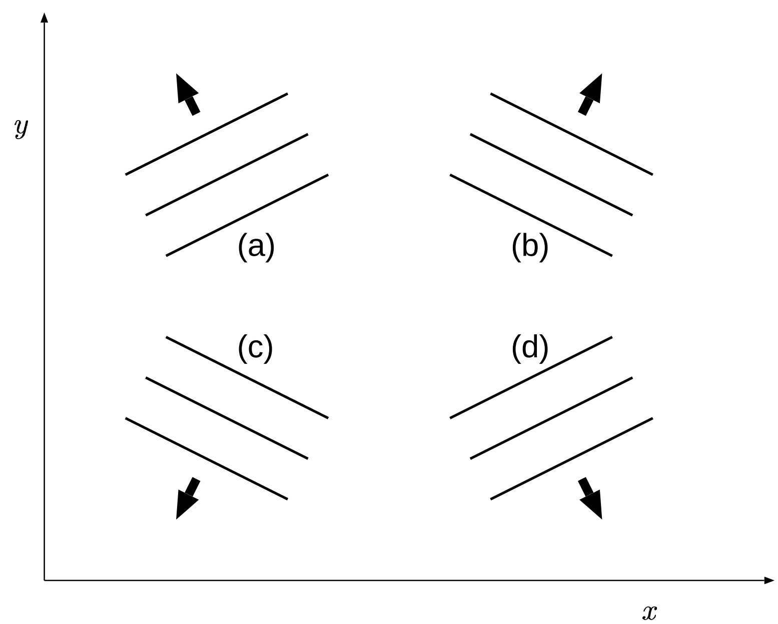


Figure 1: Four waves with phase speed toward different quadrants and identical values of $|k|$, $|l|$, $|\nu|$.

These possibilities are conventionally covered without redundancy by stipulating that $\nu > 0$, while k and l may have arbitrary sign. The generalization to three dimensions in trivial. (Other possibilities, such as demanding $k > 0$ while allowing ν and l to have arbitrary sign also eliminate the redundancy problem but are usually more awkward.)