

ATMS 101 Math Review

Winter Quarter 2004

1 Exponents

1. An *exponent* is the power to which a number (which we'll call *base*) is raised. Say we have a positive base a that is greater than zero. Then:

$$\begin{aligned}a^1 &= a && (a \text{ to the power of } 1) \\a^2 &= a \times a && (a \text{ squared}) \\a^3 &= a \times a \times a && (a \text{ cubed})\end{aligned}$$

So, the *exponents* indicate how many times the number a is multiplied by itself.

2. A number raised to the power of zero equals one. i.e.:

$$a^0 = 1$$

3. If we have a negative exponent, this means that we have a fraction with 1 in the numerator and the number raised to the *positive* exponent in the denominator. i.e.:

$$a^{-n} = \frac{1}{a^n}$$

For example, $a^{-2} = 1/a^2$.

4. If we multiply two different powers of the same number together, we end up with the number raised to the *sum of the exponents*. i.e.:

$$a^m \times a^n = a^{m+n}$$

For example,

$$\begin{aligned}a^{-1} \times a^3 &= \frac{1}{a} \times (a \times a \times a) \\&= a \times a \\&= a^2\end{aligned}$$

Note that in this example we have disguised a division as a multiplication by using negative exponents.

5. A fractional exponent is the same as a *root*. i.e.:

$$a^{1/n} = \sqrt[n]{a}$$

So that $a^{1/2}$ is the same as the square root of a , $a^{1/3}$ is the cubic root of a and so on.

6. A power of a raised to another power, is a raised to the *product of the exponents*. i.e.

$$(a^m)^n = a^{m \times n}$$

For example,

$$\begin{aligned} (a^3)^2 &= (a \times a \times a)^2 \\ &= (a \times a \times a) \times (a \times a \times a) \\ &= (a \times a \times a \times a \times a \times a) \\ &= a^6 \end{aligned}$$

This will be useful when dealing with roots since $(a^{1/n})^n = a$, so that by raising a root to the appropriate number we can get rid of it.

2 Significant figures

1. The significant figures in any number, are the digits starting from the first non-zero figure from the left. For example, in the number 0.003503 the significant figures are 3503 and in 142.50023 the significant figures are 142.50023. When presenting numbers, we will often do so “to the first n significant figures”. For example, 0.003503 written to the first 3 significant figures will be 0.00350 because 350 are the first 3 significant figures. When doing this, we have to make sure the last figure is properly rounded. For example 142.50023 written to the first 3 significant figures is 143.
2. Many times when we perform operations on numbers, our calculations will produce numbers with many many more significant figures than the numbers we started with. For example, to the question “How many times heavier is a rock of 1.1 tons than a rock of 0.7 tons” the straight forward answer is

$$\frac{1.1}{0.7} = 1.5714285714285714...$$

In this case, the significant figures extend indefinitely, but it doesn't make sense to try to recite them all to answer the question. Since the initial numbers were not given as 1.1000000000000... or 0.7000000000000..., it is apparent that the real weights were within a tenth of ton of the given values. We can't get more accuracy than what we had to begin with! An

appropriate answer may be 1.6. That is, we only considered the first two significant figures when presenting the answer. There is no clear-cut rule about how many significant figures to keep, just try to be reasonable.

3 Scientific notation

1. When working in science, we'll be dealing with numbers from a large range of scales. For example, the age of the Earth is approximately 4.6 billion years while a storm cloud may live only half an hour (or approximately 0.00006 years).

If we were to write these numbers side by side in the same units, it will look messy if we use the same degree of accuracy (number of decimal places) for both. i.e.:

$$\begin{array}{rcl} \text{Age of Earth} & = & 4600000000.00000 \text{ years} \\ \text{Life of cloud} & = & 0.00006 \text{ years} \end{array}$$

Note that age of the Earth written in this way seems much more accurate than what we really have, while the zeros on the life of a cloud provide with no information by themselves.

2. To avoid the problems mentioned, we introduce the so-called scientific notation in which only the significant (non-zero) part of the number is explicitly mentioned and the number of zeros present is indicated by a power of 10. *The exponent on the 10 tells us how many places to move the decimal point.*

Using this, we can rewrite the numbers in our previous example as:

$$\begin{array}{rcl} \text{Age of Earth} & = & 4.6 \times 10^9 \text{ years} \\ \text{Life of cloud} & = & 6 \times 10^{-5} \text{ years} \end{array}$$

To recover 4600000000 from 4.6×10^9 , we shift the decimal point in 4.6 by 9 places (the exponent on the 10) to the right. *The shift is done to the right because the exponent is positive.*

In the other case, we shift the decimal point on 6 by 5 places *to the left* because the exponent is negative.

3. When operating on numbers written in scientific notation it is useful to operate separately on the numbers before the 10 and on the powers of ten. For example, the ratio of the life of a cloud to that of the Earth is

$$\frac{6 \times 10^{-5}}{4.6 \times 10^9} = \left(\frac{6}{4.6} \right) \times \left(\frac{10^{-5}}{10^9} \right)$$

$$\begin{aligned}
&= 1 \times 10^{-5-9} \\
&= 1 \times 10^{-14}
\end{aligned}$$

The exponent on the 10 is -14 , which means that the decimal point should be moved 14 places to the left. It is a lot of zeros we are avoiding to write by using scientific notation!

4 Logarithms

1. In previous sections we've dealt with exponents. Raising a number to some power provided us with a new number. For example:

$$x = 2^3$$

yields

$$x = 8$$

In the example we were given the 2 and the 3, and we calculated the 8. What if we had been given instead the 2 and the 8, and we wanted to find out what the exponent was? In this case we'd have

$$2^y = 8$$

Our usual approach for solving algebraic equations doesn't help us. We cannot solve for y by simply adding, subtracting, multiplying or dividing both sides of the equation by any number.

So, we introduce the *logarithm*! This is just a way of expressing that we have solved for the exponent. For example, in our previous example we would write

$$y = \log_2 8$$

which simply means " y is the number by which we raise 2 to get 8". We still don't have a numerical value for y , though. That's what calculators (or numerical tables, for the old timers) are for! In general $y = \log_a b$ means " y is the number by which we raise the base a to get b ".

2. Previously we talked about logarithms with any base. However, as we've seen in previous sections, we'll be using heavily the base 10. In fact, this is so widely used, that if the base is not explicitly mentioned then it is assumed to be 10. So

$$y = \log b$$

is the same as

$$y = \log_{10} b$$

If b were 10 raised to an integer power, then y would just give the number of zeros in b . For example $\log 1000 = 3$.

- Another commonly used base is the number known plainly as e , which is a number that shows up often in mathematics and science. An approximate value for e is 2.71828, but it is an irrational number (i.e. decimals go on to infinity). The logarithms with base e are called Naperian logarithms (after John Napier) and are written

$$y = \ln b$$

which is the same as

$$y = \log_e b$$

- By considering the operation rules for exponents, we can come up with the following rules for logarithms:

- Addition/multiplication

$$\log_a b + \log_a c = \log_a (b \times c)$$

$$\text{Example: } \log_{10} 100 + \log_{10} 1000 = \log_{10} 100000$$

- Subtraction/division

$$\log_a b - \log_a c = \log_a \left(\frac{b}{c} \right)$$

$$\text{Example: } \log_{10} 100 - \log_{10} 1000 = \log_{10} 0.1$$

- Exponent to multiplication

$$\log_a b^n = n \log_a b$$

$$\text{Example: } \log_{10} 2^3 = 3 \log_{10} 2$$

- Change of base

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\text{Example: } \log_{100} 1000 = \frac{\log_{10} 1000}{\log_{10} 100} = \frac{3}{2}$$

5 Direct and inverse proportionality

- We usually say that two quantities are directly proportional when a change in one of them implies a change in the other by the same factor. For example, the doubling in one of them implies a doubling of the other one. Generally this is indicated as $x \propto y$, which says “ x is directly proportional to y ”.

An example from physics is Hooke’s law for a spring, which says that the force required to extend a spring is proportional to the length by which it

is extended. So, the more you want to extend the spring, the greater the force you need to apply. In mathematical notation this can be written as:

$$F = kx$$

where F is the force applied, x the extension and k is called “spring constant”, but in general it is the “proportionality constant” between F and x .

Note that the proportionality constant could be negative, so that an *increase* in one of the quantities could imply a *decrease* in the other. An example can be found in energy transfer between two bodies A and B. Since energy is conserved, what is lost by one is gained by the other. If 5 calories are transferred from A to B, the change of the energy of A is -5 calories and the change in energy of B is +5 calories. If 10 calories were transferred, the respective changes would be -10 and 10 respectively. Therefore, the changes in energy of both bodies are proportional and the proportionality constant is -1.

2. Two quantities are said to be inversely proportional if the product of the two are constant. So, if one of the quantities is doubled then the other is halved, if one is tripled then the other is reduced to a third, and so on.

A physical example is given by Boyle’s law constant temperature, which says that the product of the pressure and volume of a gas at constant temperature is constant . In mathematical form this can be written as

$$PV = \text{Constant}$$

So, if the volume V of the gas is reduced to half, then the pressure P is doubled. This can also be written as $P = \text{Constant} \times \frac{1}{V}$, which tells us the fact that the two quantities are inversely proportional to each other is equivalent to saying that one of them is proportional to 1 divided by the other. Using the notation we introduced previously, if x and y are inversely proportional, then $x \propto 1/y$.