Atmospheric Science 536—Winter 2010

**Homework 2**

1. Our starting point is the inviscid hydrostatic Boussinesq equations on an $f$-plane

$$\frac{du}{dt} - fv + \frac{\partial P}{\partial x} = 0,$$

$$\frac{dv}{dt} + fu + \frac{\partial P}{\partial y} = 0,$$

$$\frac{\partial P}{\partial z} = b,$$

$$\frac{db}{dt} + N^2 w = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where

$$b = g\frac{\theta - \theta^*}{\theta_0}, \qquad P = c_p\theta_0(\pi - \pi^*), \quad \text{and} \quad N^2 = \frac{g}{\theta_0}\frac{\partial \theta^*}{\partial z},$$

$\theta^*(z)$ and $\pi^*(z)$ are vertically varying reference state values in hydrostatic balance, $\pi = (p/p_0)^{R/c_p}$, and $\theta_0$, $p_0$ and $f$ are constants.

Your goal is to explore the behavior of *x-independent* small-amplitude perturbations to a zonal flow with shear. Let the basic-state have the functional form $\overline{u}(y,z)$, $\overline{P}(y,z)$, $\overline{b}(y,z)$ in hydrostatic and geostrophic balance, and assume that $\overline{u}$ and $\overline{b}$ vary *linearly* with $y$ and $z$ (i.e., $\overline{u}(y,z) = u_0 + \overline{u}_y y + \overline{u}_z z$). Note that these basic state variables are defined without having any particular relation to $\theta^*(z)$ and $\pi^*(z)$.

(a) Show that as a consequence of the basic state being in hydrostatic and geostrophic balance,

$$f\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}\,.$$

This is thermal wind balance.

(b) Show that *x-independent* small-amplitude perturbations about this basic state statisfy the system of equations

$$\frac{\partial u'}{\partial t} + v'\frac{\partial \overline{u}}{\partial y} + w'\frac{\partial \overline{u}}{\partial z} - fv' = 0,$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{\partial P'}{\partial y} = 0,$$

$$\frac{\partial P'}{\partial z} = b',$$

$$\frac{\partial b'}{\partial t} - v' f\frac{\partial \overline{u}}{\partial z} + N_s^2 w' = 0,$$

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

where

$$N_s^2 = N^2 + \frac{\partial \bar{b}}{\partial z}.$$

(c) Define a streamfunction for flow in the N-S plane

$$v' = -\frac{\partial \psi}{\partial z}, \qquad w' = \frac{\partial \psi}{\partial y},$$

and show that the preceding linear system reduces to the single equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \psi}{\partial z^2}\right) + \left(N_s^2 \frac{\partial^2 \psi}{\partial y^2} + F^2 \frac{\partial^2 \psi}{\partial z^2} + 2S^2 \frac{\partial^2 \psi}{\partial y \partial z}\right) = 0,$$

where

$$F^2 = f\left(f - \frac{\partial \overline{u}}{\partial y}\right), \qquad S^2 = f\frac{\partial \overline{u}}{\partial z}.$$

2. Assume a solution of the form

$$\psi = e^{i\sigma t} e^{ik(y \sin\phi - z\cos\phi)}.$$

(a) What is the physical significance of $\phi$ ?

(b) Determine an expression for $\sigma^2$ in terms of $N_s^2$, $F^2$, $S^2$, and $\phi$.

(c) For the expression derived in (b), the minimum value of $\sigma^2$ with respect to $\phi$ occurs when

$$\tan\phi = \frac{S^2}{N_s^2}.$$

You *do not* need to show thus, instead use this fact to determine the mathematical condition under these perturbations will be unstable. Relate your result to the Ertel potential vorticity of the basic-state flow.

(d) What is the name of the type of instability that you have just analyzed? *Hint: it has been discussed from the parcel standpoint in class.*

*Due Thursday February 4th*