Atmospheric Science 536—Winter 2010

Homework 3

1. As discussed in class, the Fourier transform $\mathcal{F}$ of a *real-valued* function $\phi(x)$ may be defined as

$$\hat{\phi}(k) = \mathcal{F}[\phi(x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x)e^{-ikx}\,dx.$$

The inverse transform may be defined as

$$\phi(x) = \mathcal{F}^{-1}[\phi(k)] = \mathrm{Re}\int_{0}^{\infty} \hat{\phi}(k)e^{ikx}\,dk.$$

Consider 2D Boussinesq flow in a basic state with constant N and U over a "Witch of Agnesi" mountain

$$h(x) = \frac{h_0 a^2}{x^2 + a^2}.$$

To learn why it's called a "witch," see

`http://en.wikipedia.org/wiki/Witch_of_Agnesi`

(a) Let $\ell = N/U$ be the Scorer parameter. Using the lecture notes (you don't need to re-derive results obtained in lecture) and the fact that $\hat{h}(k) = h_0 a e^{-a|k|}$, show that

$$w(x,z) = \mathrm{Re}\int_{0}^{\infty} i\left[h_0 U k a e^{-ak}\right] e^{i(\ell^2-k^2)^{1/2}z} e^{ikx}\,dk. \tag{1}$$

Assuming $U > 0$, explain how this expression includes the physically appropriate upper boundary condition.

(b) That portion of w that is independent of x and z is given by the factor in square brackets in the preceding. At what wavenumber k is this factor maximized?

2. One may obtain an easier integral than that in (1) by recasting the governing equation for the displacement of a streamline about its elevation in the undisturbed upstream flow δ. In the linear limit, $w = U\partial\delta/\partial x$.

(a) Using the results you obtained in Problem 1, derive an integral relation similar to that in (1) for $\delta(x,z)$.

(b) Explain how for "wide" mountains ($a \gg b \gg 1/\ell$), the integral in derived in (2a) is approximately

$$\delta(x,z) = \mathrm{Re}\int_{0}^{\infty} h_0 a \exp[i(kx + \ell z) - ak]\,dk. \tag{2}$$

The assumption ($a \gg b \gg 1/\ell$ requires a to be at least 2 orders of magnitude bigger than $1/\ell$ (to be able to squeeze b into the inequality chain). This assumption is easily satisfied in the hydrostatic limit. *Hint:* Break the integral into two parts, one from 0 to $1/b$ and one from $1/b$ to ∞ and argue that one of these is small. Use $a \gg 1/\ell$ to simplify the integrand in the other and then argue that it can, to good approximation, be extended to the integral from 0 to ∞.

(c) Show that (2) evaluates to

$$\delta(x,z) = h_0 a \left(\frac{a \cos \ell z - x \sin \ell z}{x^2 + a^2} \right).$$

3. Now consider small-amplitude internally stratified Boussinesq flow over a ridge. Assume N and U are constant.

(a) Let $\delta(x,z)$ be the displacement of a streamline about its elevation in the undisturbed upstream flow. Derive the expression for the perturbation horizontal velocity u in terms of δ for small amplitude Boussinesq flow.

(b) Use the expression you derived in (2c) for the displacements generated by flow with constant N and U over a Witch of Agnesi mountain, to explain how the parameter Nh_0/U can be straightforwardly interpreted as a measure of the nonlinearity of this type of cross-mountain flow.

(c) Generalize the argument in (b) by showing that Nh_0 is a reasonable scale for the perturbation horizontal velocity in steady small-amplitude mountain waves generated by flow over a long ridge of arbitrary shape. (Here h_0 is the maximum height of the topography). *Hint:* Use scaling arguments; let L be a characteristic horizontal length scale, D the vertical length scale, and connect u with w via the continuity equation.

Due Thursday February 18th