

Chapter 3

3.1 Basic Equations in isobaric coordinates (x,y,p,t)

Work through the derivations in this section on your own, in class we will focus on solving problems and building intuition. All derivations are straightforward and make good practise, except S_p is rather tricky (even though Holton says “it is easy to show”).

The horizontal momentum equation in isobaric coordinates is

$$\frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla_p \phi \quad (1)$$

where $\mathbf{V} = u\hat{i} + v\hat{j}$ is the horizontal velocity and the little subscript p means holding pressure constant. The total derivative written out in component form is coordinate system dependent. Even though this is only the *horizontal* momentum equation, D/Dt still depends on vertical advection:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

where $\omega = Dp/Dt$ is the vertical velocity component in isobaric coordinates. ω has the opposite sign of w .

The geostrophic wind in isobaric coordinates is independent of ρ

$$f\mathbf{V}_g = \hat{k} \times \nabla_p \phi$$

For homework this week you get to show an additional extremely nice property of \mathbf{V}_g in isobaric coordinates — it is divergentless on pressure surfaces when $f=\text{constant}$:

$$\nabla_p \cdot \mathbf{V}_g = 0$$

The continuity equation also is independent of ρ (and this form is not restricted to an incompressible flow)

$$\nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0 \quad (2)$$

The thermodynamics energy equation in isobaric coordinates is

$$\partial T / \partial t + \mathbf{V} \cdot \nabla_p T - S_p \omega = J / c_p \quad (3)$$

where the static stability in isobaric coordinates is

$$S_p = -\frac{T}{\theta} \frac{\partial \theta}{\partial p} = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \frac{\Gamma_d - \Gamma}{g\rho}.$$

Compare with the z-coordinate equation

$$S = \frac{T}{\theta} \frac{\partial \theta}{\partial z} = \Gamma_d - \Gamma.$$

For reasonably small vertical displacements we can usually approximate $\Gamma \approx \text{constant}$, hence $S \approx \text{constant}$. However S_p is not approximately constant because ρ varies exponentially. This is a disadvantage of isobaric coordinates.

3.2 Balanced Flow

Balanced flows follow relatively simple force balances.

Here we let $\omega = 0$ and only consider flow that are approximately in the horizontal plane. ω resurfaces again near the end of this chapter.

3.2.1 Natural Coordinates

\hat{t} = tangent to *velocity* at each instant

\hat{n} = normal to *velocity* at each instant

s is the distance along the parcel's path

$\mathbf{V} = V\hat{t}$ is the velocity

$V = Ds/Dt$ is the speed

$$\frac{D\mathbf{V}}{Dt} = \hat{t} \frac{DV}{Dt} + V \frac{D\hat{t}}{Dt}$$

Fig 3.1 from Holton illustrates the natural coordinate system.

From the figure you can see that R is the radius of curvature and $\frac{D\hat{t}}{Dt}$ is an angular velocity with direction normal to the path:

$$\frac{D\hat{t}}{Dt} = \frac{V}{R}\hat{n}.$$

Hence, the momentum equation in natural coordinates is

$$\frac{D\mathbf{V}}{Dt} = \hat{t}\frac{DV}{Dt} + \hat{n}\frac{V^2}{R} \quad (4)$$

The first term is the acceleration along the path and the second is the centripetal acceleration *due to curving relative motion on Earth in the horizontal plane* (different from the centrifugal force swept into g due rotation rate of Earth).

The Coriolis force in natural coordinates is

$$-f\hat{k} \times \mathbf{V} = -fV\hat{n}.$$

The pressure gradient force (PGF) in natural coordinates is

$$-\nabla\Phi = -\hat{t}\frac{\partial\Phi}{\partial s} - \hat{n}\frac{\partial\Phi}{\partial n}$$

Setting the acceleration (Eq. 4) equal to the sum of Coriolis and PGF gives two component equations (no vector symbols now in component equations)

$$\frac{DV}{Dt} = -\frac{\partial\Phi}{\partial s} \quad \hat{t}\text{--component} \quad (5)$$

$$\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n} \quad \hat{n}\text{-component} \quad (6)$$

Eq (5) states the parcel acceleration along the parcel path equals the PGF along the parcel path. Eq (6) states the accelerations normal to the parcel path (which is the centripetal plus Coriolis) equal the PGF normal to the parcel path. Generally the two accelerations are moved to the right hand side and referred to as forces and then Eq ?? becomes $0 = C_e + C_o + PGF_n$, where the right hand side is the sum of three forces: centrifugal, Coriolis and pressure gradient normal to the flow.

By definition, $V > 0$ but R can be positive or negative: $R > 0$ for cyclonic flow and $R < 0$ for anticyclonic or antibaric (clockwise flow around a low) flow

3.2.2 Geostrophic Flow

If $|R| \rightarrow \infty$ then $\mathbf{V} = \mathbf{V}_g$ are both in the \hat{t} direction and $V_g \equiv -f^{-1}\frac{\partial\Phi}{\partial n}$, is found by equating the Coriolis force and PGF.

For any R , if $DV/Dt = 0$ then $\mathbf{V} \neq \mathbf{V}_g$, but we can still define a geostrophic flow: $V_g \equiv -f^{-1}\frac{\partial\Phi}{\partial n}$. It is locally parallel to height contours and hence is either parallel or antiparallel to \mathbf{V} .

For any R , if $DV/Dt \neq 0$ then we can still define a geostrophic flow

$$f\hat{k} \times \mathbf{V}_g = -\nabla\Phi$$

but this time you can expect a nonzero angle between \mathbf{V}_g and \mathbf{V} .

In all three cases, the PGF is to the left of \mathbf{V}_g . In the first two the PGF along the path is zero, so the math is easier.

3.2.3 Inertial Flow

When $\partial\phi/\partial n = 0$ the flow is called “inertial” and the remaining balance of centrifugal and Coriolis forces in Eq. (6) yields a circular flow with $R = -V/f$. $R < 0$ always for inertial flow, so the motion is clockwise in the northern hemisphere.

3.2.4 Cyclostrophic Flow

If horizontal scales are small enough to neglect the Coriolis force, then Eq. (6) can be written

$$\frac{V^2}{R} = -\frac{\partial\Phi}{\partial n}$$

and

$$V = \left(-R\frac{\partial\Phi}{\partial n}\right)^{1/2}.$$

The centrifugal force C_e always points away from the center of rotation, so the PGF must always point towards it. Remember V is always positive, but nothing requires the flow to be clockwise or counterclockwise. Hence it can be either. The “normal” direction is always to the left of the direction of flow.

3.2.5 Gradient flow

Gradient flow is a special case when $DV/Dt = 0$ so V is time-independent and always flows parallel to lines of constant geopotential, defined by Eq (6):

$$V = \frac{-fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial\phi}{\partial n}\right)^{1/2} = \frac{-fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{1/2} \quad (7)$$

V must be real so the argument of the square root must be positive. This is only an issue for anticyclonic flow around a high because $R < 0$ and $V_g > 0$

$$fV_g = -\frac{\partial\phi}{\partial n} < \frac{-Rf^2}{4}. \quad (8)$$

This requirement makes pressure gradients flat at the center of highs.

In class exercise: Compute the gradient wind speed

$$V = \frac{-fR}{2} \pm \left(\frac{f^2 R^2}{4} - R \frac{\partial\phi}{\partial n} \right)^{1/2}$$

for the following cases (all with $f = 10^{-4} \text{ s}^{-1}$)

1. a regular low with $-\partial\phi/\partial n = 0.86 \times 10^{-3} \text{ m/s}^2$ and $R = 250 \text{ km}$.
2. an anomalous low with $-\partial\phi/\partial n = -0.86 \times 10^{-3} \text{ m/s}^2$ and $R = -250 \text{ km}$.
3. a regular high with $-\partial\phi/\partial n = 0.26 \times 10^{-3} \text{ m/s}^2$ and $R = -250 \text{ km}$.
4. an anomalous high with $-\partial\phi/\partial n = 0.26 \times 10^{-3} \text{ m/s}^2$ and $R = -250 \text{ km}$.
5. Is Equation 4 violated for a high if $-\partial\phi/\partial n = 0.86 \times 10^{-3} \text{ m/s}^2$ and $R = -250 \text{ km}$.
6. What will the wobbly path of the actual parcel motion look like if the low is changed to a high? (Do not make the gradient wind approximation. Just be qualitative.)

3.4 Thermal Wind

“The geostrophic wind must have vertical shear in the presence of a horizontal temperature gradient,” (Holton p70) Think carefully about what this means.

Mathematically it can be written

$$\frac{\partial \mathbf{V}_g}{\partial z} \neq 0 \quad \text{if} \quad \nabla T \neq 0$$

Consider a special case of motion parallel to the y-axis (into the page for the axes below).

In this case isobars only depend on x and z.

Recall that

$$\mathbf{V}_g = \frac{1}{f} \hat{k} \times \nabla_p \Phi$$

Then for our special case (recall $\Phi = gz$):

$$v_g = \frac{g}{f} \left. \frac{\partial z}{\partial x} \right|_p,$$

which says the local slope of an isobar $p(x, z)$ determines the geostrophic wind speed.

The geostrophic wind must have vertical shear if the slopes of isobars vary with height. When isobars vary in this way, the vertical separation of isobars (or thickness) increases with x . Recall that the hypsometric equation gives the thickness of layers:

$$\delta z = \frac{R}{g} < T > \ln \frac{p_1}{p_0}$$

At right, δz increases with x because $< T >$ increases with x , so v_g increases with height too.

Because “thickness” δz is proportional to temperature $< T >$:

v_g must vary with height whenever T varies in x .

u_g must vary with height whenever T varies in y .

Technically the “thermal wind” equation is

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f} \hat{k} \times \nabla_p T$$

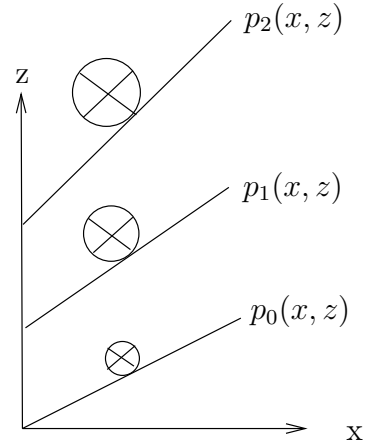
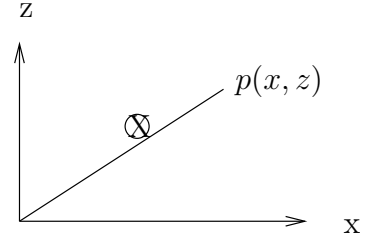
which relates vertical shear of the geostrophic wind to temperature gradients.

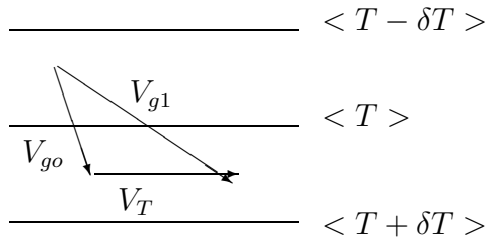
To solve problems we usually first integrate the thermal wind equation and define the “thermal wind”

$$\mathbf{V}_T \equiv \mathbf{V}_g(p_1) - \mathbf{V}_g(p_0) = -\frac{R}{f} (\hat{k} \times \nabla_p < T >) \ln \frac{p_1}{p_0} \quad (9)$$

where $< T >$ is the average temperature in the layer. Also

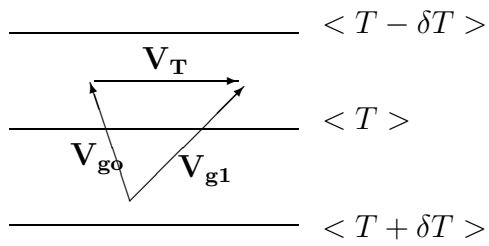
$$\mathbf{V}_T = -\frac{1}{f} \hat{k} \times \nabla_p (\Phi_1 - \Phi_0) \quad (10)$$





Winds are backing = counterclockwise rotation with increasing height.

Causes cold advection, expect temperatures to fall.



Winds are veering = clockwise rotation with increasing height.

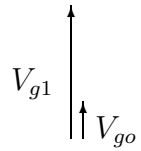
Causes warm advection, expect temperatures to rise.

All that is needed is 2 of the 3 vectors to solve problems. Or because knowing $\nabla < T >$ gives us V_T , if $< T >$ is known, then the geostrophic wind can be found at a given level provided the geostrophic wind is known at another level.

The average temperature advection in the layer can be estimated from the advection of $< T >$ by $< V_g >$.

Sample Problem

Seattle at 6:30AM



For $p_o = 1000$ hPa and $p_1 = 800$ hPa,
 $V_{g0} = 0$ m/s and $V_{g1} = 6$ m/s to the north.
 Ignore friction and topography. What is $\nabla_p < T >$
 and what is the temperature advection?

Seattle at 10:30AM

$\mathbf{V}_{g0} = 4 \text{ m/s } \hat{j}$ and $\mathbf{V}_{g1} = 2 \text{ m/s } \hat{i} + 6 \text{ m/s } \hat{j}$
 Ignore friction and topography. What is $\nabla_p < T >$
 and what is the temperature advection?

3.4.1 Barotropic and Baroclinic Atmospheres

Barotropic means $\rho = \rho(p)$. Hence isobars are constant density surfaces.

In the ideal gas approximation, $T = p/\rho R$ so $T = T(p)$. Hence isobars in a barotropic atmosphere have constant temperature too: $\nabla_p(T) = 0$, so \mathbf{V}_g has no vertical shear. When $\mathbf{V} \approx \mathbf{V}_g$ in a barotropic atmosphere, V doesn't vary much in height either.

Baroclinic means $\rho = \rho(p, T)$, which is more often the case, so generally V does vary with height.

3.5 Vertical Motion

The scale of vertical motions is $W \sim 1$ cm/s

Weather balloon soundings and satellites only measure to an accuracy of about 1m/s, so w must be found indirectly. In this section we will see two not very successful ways to estimate w as a residual. Both use isobaric coordinates, so first we must find a relationship between ω and w . (Notation note: I'm switching from w to w in this section to denote the vertical velocity Dz/Dt because I find w hard to distinguish from ω . This is the only time I'll do it though.) Recall:

$$\omega \equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla_z p + w \frac{\partial p}{\partial z}$$

V_g is normal to ∇p , so $\mathbf{V}_g \cdot \nabla p = 0$. Defining the ageostrophic wind

$$\mathbf{V}_a \equiv \mathbf{V} - \mathbf{V}_g \tag{11}$$

along with hydrostaticity in the last term gives

$$\omega = \frac{\partial p}{\partial t} + \mathbf{V}_a \cdot \nabla_z p - w g \rho$$

Using scale analysis we find

$$\frac{\partial p}{\partial t} \sim \frac{\delta p U}{L} \sim 10 \text{ hPa/d}$$

where $L/U \sim 10^5 \text{ s} \sim 1 \text{ d}$ is the time scale (see Holton p 39) and $\delta p \sim 10 \text{ hPa}$ is the change in pressure from one synoptic system to the next. The next two terms are about 1 hPa/d and 100hPa/d. Hence

$$\omega \approx -w g \rho \quad \textbf{away from the surface.}$$

This relation is not appropriate next to the surface where $w_s = 0$ (no flow through the surface), but $\omega_s \neq 0$ in general. Instead take the next order term:

$$\omega_s \approx \frac{\partial p_s}{\partial t} \quad \textbf{near the surface.} \quad (12)$$

Incidentally, both w and ω are zero at the top of the atmosphere.

3.5.1 Kinematic Method (aka non predictive)

Continuity in isobaric coordinates is

$$\nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0$$

Integrating we get

$$\omega(p) - \omega(p_s) = - \int_{p_s}^p \nabla_p \cdot \mathbf{V} dp \quad (13)$$

In homework 4 you showed $\nabla_p \cdot \mathbf{V}_g = 0$ for constant f . The integrand is then roughly $\nabla_p \cdot \mathbf{V}_a$.

But V_a cannot be measured accurately, so this method is not recommended.

3.5.2 Adiabatic Method

The thermodynamic energy equation (Eq. 3) can be written

$$\omega = S_p^{-1} \left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T - J/c_p \right)$$

- The last term is usually negligible.

- The middle term can be approximated $\mathbf{V}_g \cdot \nabla T$ using a snapshot of T and Z or ϕ .
- The first term is non negligible and difficult to measure. T has a large daily cycle plus it has variability from system to system, yet it is only measured a couple of times daily. This is known as aliasing, and it is very inaccurate. So this method is not recommended either.

3.6 Surface Pressure Tendency

Recycle Eq (13) but take $p = 0$ (ie the top of the atmosphere) so $\omega(0) = 0$. Hence

$$\omega(p_s) = \int_{p_s}^0 \nabla_p \cdot \mathbf{V} dp \quad (14)$$

Substituting from Eqs. 11 and 12

$$\frac{\partial p_s}{\partial t} \approx - \int_0^{p_s} \nabla_p \cdot \mathbf{V}_a dp \quad (15)$$

Even though $\nabla_p \cdot \mathbf{V}_a$ is hard to measure accurately, Eq (15) gives a useful relation for qualitative understanding.

The four primitive equations in isobaric coordinates in isobaric coordinates can be written:

1) Horizontal momentum

$$\frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla_p \phi$$

2) Continuity

$$\nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial z} = 0$$

3) Thermodynamics energy

$$\partial T / \partial t + \mathbf{V} \cdot \nabla_p T - S_p \omega = J / c_p$$

4) Hydrostaticity

$$\frac{\partial \Phi}{\partial p} = -\alpha$$

These four equations together can be used as a complete predictive model with the addition of a lower boundary condition for Φ_s . Note that $w = Dz/Dt = 0$ means:

$$\frac{\partial \Phi_s}{\partial t} = -\mathbf{V}_a \cdot \nabla \Phi_s - \omega \frac{\partial \Phi_s}{\partial p}.$$

Neglecting the advection by the ageostrophic wind and using Eq (14) and hydrostaticity gives:

$$\frac{\partial \Phi_s}{\partial t} \approx -\frac{RT}{p_s} \int_0^{p_s} (\nabla \cdot \mathbf{V}) dp.$$

These equations are too hard to solve on paper for typical atmospheric flows, but numerical weather/climate prediction models use them routinely.