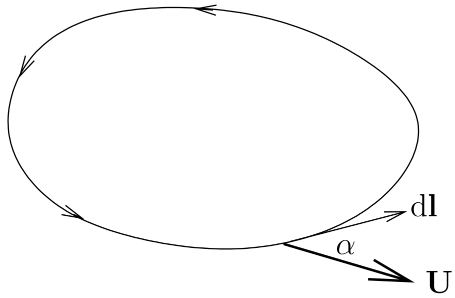
Chapter 4

4.1 The Circulation Theorem

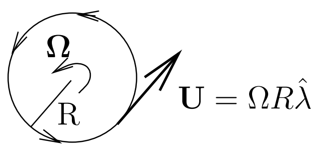
Circulation is a measure of rotation. It is calculated for a closed contour by taking the line integral of the velocity component tangent to the contour evaluated along the contour:



$$C \equiv \oint \mathbf{U} \cdot d\mathbf{l} = \oint |\mathbf{U}| \cos\alpha \; dl.$$

By convention, $C > 0$ for counterclockwise flow, hence the contour must be counterclockwise.

Consider the special case of a circular ring with radius R rotating with constant angular velocity $\mathbf{\Omega}$ as shown. Note that Ω and R here are not for the rotation rate and radius normal to Earth's axis of rotation.



They are defined as in the figure. Also note $\hat{\lambda}$ is a unit vector tangent to the circle, so $d\mathbf{l} = R \; d\lambda \; \hat{\lambda}$. For this special case where $\mathbf{U}$ and $d\mathbf{l}$ are in the same direction, so $C = \int_0^{2\pi} \Omega R^2 d\lambda = 2\pi\Omega R^2$.

Note that ΩR^2 is the angular momentum about the axis of rotation.

The derivation of the circulation theorem begins with the momentum equation without friction:

$$\frac{D_a \mathbf{U}_a}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^*.$$

Recall true gravity $g^* = g - \Omega^2 \mathbf{R}$. Holton forgets the star in his derivation. Fortunately the end result is the same.

Taking the contour integral of the momentum equation, with $g = -\nabla\Phi$ we arrive at:

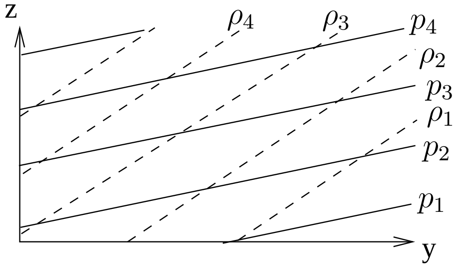
$$\oint \frac{D_a \mathbf{U}_a}{Dt} \cdot d\mathbf{l} = \oint \frac{\nabla p}{\rho} \cdot d\mathbf{l} - \oint \nabla\Phi \cdot d\mathbf{l} - \oint \Omega^2 \mathbf{R} \cdot d\mathbf{l}.$$

Intuitively, the last term must be zero because $\mathbf{R}$ is a position vector and therefore does not "circulate". The rest of the derivation can be found in Holton to end with Kelvin's

Circulation Theorm

$$\frac{DC_a}{Dt} = \frac{D}{Dt}\oint \mathbf{U}_a \cdot d\mathbf{l} = -\oint \rho^{-1} dp$$

The term $-\oint \rho^{-1} dp$ is called the "solenoid" term. The word solenoid refers to the parallelograms that form when isobars and lines of constant density are drawn together for a baroclinic atmosphere.



Consider a line integral around one of the parallelograms for a baroclinic atmosphere. Clearly the solenoid term is nonzero provided the parallelogram is finite.

Earth dwellers usually prefer to work with the circulation relative to Earth's rotation: $C = C_a - C_E$, where the circulation due to Earth's rotation Ω is C_E. For a contour on Earth's surface with area A, $C_E = 2\Omega A < \sin\phi >$ where $A < \sin\phi >$ is the area of the contour projected onto the equatorial plane (see Holton Fig 4.2 to visualize) and $\phi =$ latitude. Hence, we arrive at the Bjerknes Circulation Theorem

$$\frac{DC}{Dt} = -\oint \rho^{-1} dp - 2\Omega \frac{D(A < \sin\phi >)}{Dt}$$

where Ω is Earth's rotation rate.

For a barotropic atmosphere the density surfaces are also isobaric surfaces, so the parallelograms in the figure above are infinite and the solenoid term is zero.

There are a couple of good examples in Holton that are worth working through.

4.2 Vorticity

Circulation describes the average flow over a finite area defined by the line integral. Hence it is a "macroscopic" measure of rotation. In contrast vorticity is a "microscopic" measure of the rotation. Vorticity is a field variable (it may be unique at every point).

Absolute and relative vorticity are defined

$$\omega_a \equiv \nabla \times \mathbf{U}_a \qquad \omega \equiv \nabla \times \mathbf{U}.$$

In Cartesian coordinates

$$\nabla \times \mathbf{U} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & v & \mathrm{w} & u & v \end{array} \right|$$

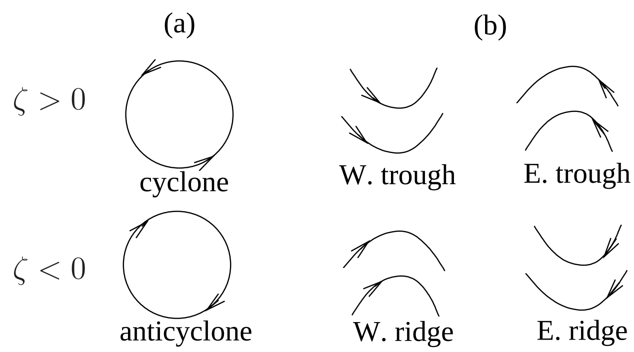
(add the full diagnals that slope down to the right and subtract those that slope down to the left)

$$= \left(\frac{\partial \mathrm{w}}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial \mathrm{w}}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

The vertical components are assigned their own variable names "eta" and "zeta"

$$\eta \equiv \hat{k} \cdot \nabla \times \mathbf{U}_a \qquad \zeta \equiv \hat{k} \cdot \nabla \times \mathbf{U}.$$

Their difference is the planetary vorticity or $f = \eta - \zeta$ (also the Coriolis parameter), which is the local vertical component of vorticity due to Earth's rotation.



Think back to the worksheet on circulation where you showed that $C = -L^2 \partial u/\partial y$ for an easterly windfield that increased linearly to the north. By extension, if there is also a northerly component of the wind that increases linearly to the east, then $C = L^2(\partial v/\partial x - \partial u/\partial y) = L^2\zeta$. If we let the sides of the square approach zero length, then it is easy to show (see Holton) that

$$\zeta = \lim_{A \to 0} C/A$$

where A = area inside the contour.

From Stoke's theorem

$$\oint \mathbf{U} \cdot d\mathbf{l} = \int\int_A (\nabla \times \mathbf{U}) \cdot \mathbf{n} \, dA$$

where if the area is in the horizontal plane then **the average ζ for an area A is**

$$C = \int\int_A \bar{\zeta} \, dA$$

Example: This is easy to see for a disk in solid-body motion with angular velocity ω, where $C = 2\pi\omega R^2$ at radius R, then $\bar{\zeta} = C/\pi R^2 = 2\omega$.

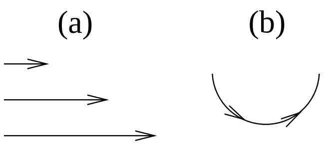
Final note: For the special cases of a disk in solid-body motion and the square around a wind field that varies linearly with distance, the vorticity at a point is equal to the average vorticity over the area, $\zeta = \bar{\zeta}$. This is not true in general however. For example, a typical cyclone has decreasing ζ near the center.

4.2.1 Vorticity in natural coordinates

Returning to the natural coordinates as in Ch 3:

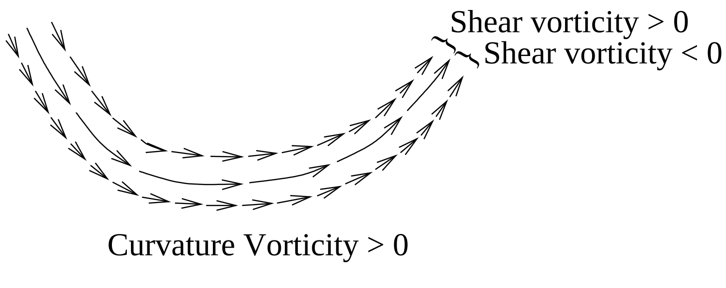
$$\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

The first term is called the shear vorticity and the second the curvature vorticity.



The figure illustrates (a) shear vorticity and (b) curvature vorticity. Remember that $\hat{n}$ points to left of the wind, so $-\partial V/\partial n > 0$ in (a), and R_s is positive so $V/R_s > 0$ in (b).

For a flow as shown, there is a combination of shear and curvature vorticity.



The subscript s on R_s indicates radius of curvature of a *streamline* rather than the parcel trajectory. The streamline is an instantaneous snapshot of motion while a trajectory concerns a finite length of time. If the pressure field is not time dependent, then there is no difference. Of course generally storm systems move eastward, so steamlines and trajectories are very different (more about this soon).

4.3 Potential Vorticity

Not only is the solenoid term zero in a barotropic atmosphere, it is also zero on isentropic surfaces (constant θ). Flow following isentropics surfaces is adiabatic. We are motivated to follow flow on isentropic surfaces because the circulation theorem is much simpler without the solenoid term.

Start with Poisson's equation $\theta = T(p_s/p)^{R/c_p}$ and replace T with ρ using the ideal gas law and $c_v = c_p - R$, we find

$$\rho = (R\theta)^{-1} p_s^{R/c_p} p^{c_v/c_p}.$$

Because $\rho = \rho(p)$ for constant θ, the solenoid term vanishes for adiabatic flow. The circulation theorem is then

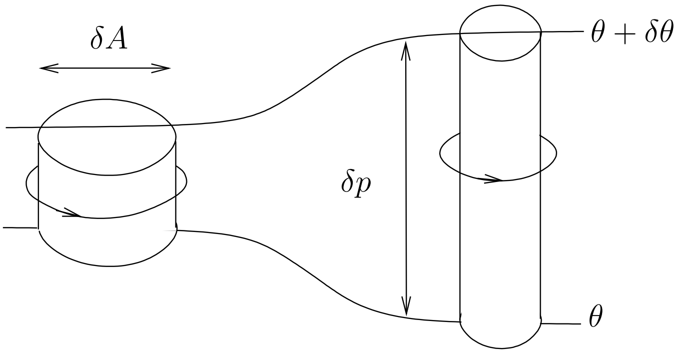
$$\frac{D}{Dt}(C + 2\Omega \sin\phi \delta A) = 0$$

where we are taking only fairly small contours. If θ-surfaces are approximately horizontal so $f = 2\Omega \sin\phi$ and $\zeta \approx C/\delta A$, then

$$\frac{D}{Dt}[(\zeta_\theta + f)\delta A] = 0$$

and following the motion

$$(\zeta_\theta + f)\delta A = \text{Const}$$



The mass of a parcel is also constant following the motion, so our vortex, which is also a parcel, starts out stubby and wide when the isentropes are close and stretches tall and thin as the isentropes widen. If the depth of the vortex is "mass weighted" $-\delta p/g$, then $\delta A \propto (-g/\delta p)$ as illustrated.

Between isentropes $\delta\theta$ = constant, so $\delta A \propto (-g\delta\theta/\delta p)$ too. Hence as $\delta p \to 0$ following the motion the quantity

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const}$$

P is Ertel's potential vorticity, or isentropic PV.

For the special case of a homogeneous, incompressible fluid (constant ρ) then we need not mass weight, and the "depth" of the fluid is simply h. Hence $\delta A \propto h^{-1}$ and following the motion

$$(\zeta_\theta + f)/h = \text{Const}$$

For the really special case of a homogeneous, incompressible fluid with constant depth $(\zeta_\theta + f) = \text{Const}$. But this is a highly idealized case.

4.4 Vorticity Eq

4.4.1 Cartesian Coordinates

Because $\zeta = \hat{k} \cdot \nabla \times V$,

$$\frac{\partial}{\partial x}\,[\text{v} - \text{momentum eq. in Cartesian Coordinates}]$$

$$\text{minus}\;\; \frac{\partial}{\partial y}\,[\text{u} - \text{momentum eq. in Cartesian Coordinates}]$$

gives

$$\frac{D(\zeta + f)}{Dt} = \text{ vorticity dilution } + \text{ tilting horizontal vortices } + \text{ microscopic solenoid}$$

- Vorticity dilution, or the divergence term, $-(\zeta + f)\nabla \cdot \mathbf{V}$, is due to stretching (vorticity creation) or shrinking (vorticity dilution) of vortex tubes. It is analogous to the skater who stretches out his/her arms or holds them straight up while spinning. In other words, the divergence term is the fluid analog of the conservation of angular momentum for a rigid body. Picture a purely convergent fluid that acquires rotation from the Coriolis force. The divergence term is most effective at creating vorticity in extratropical cyclones and along fronts. It probably should be called the convergence term since it appears with a minus sign, but it hard to change tradition.

- Tilting horizontal vorticies is the transformation of vertical shear in the horizontal wind into vertical vorticity. The vertical shear in the horizontal wind creates a horizontal vortex tube and a horizontal gradient of the vertical wind tips the horizontal vortex tube so it develops a vertical component. This term is important in frontogenesis and supercell thunderstorms.

- Microscopic solenoid is like the solenoid term in the circulation theorem but in the limit that the circulating area or cell goes to zero. Imagine the sea breeze or Hadley cell circulation with at least a partial projection in the horizontal plane. This term is not significant for synoptic scale motions.

4.4.2 Isobaric Coordinates

In isobaric coordinates $\zeta = \hat{k} \cdot \nabla_p \times V$. Following the same method as above except using isobaric u and v momentum eq., there is no solenoid term in the vorticity equation:

$$\frac{D(\zeta + f)}{Dt} = -(\zeta + f)\nabla \cdot \mathbf{V} + \hat{k} \cdot \left(\frac{\partial \mathbf{V}}{\partial p} \times \nabla \omega \right)$$

Cartesian and isobaric ζ's differ slightly from loss of solenoid term in the isobaric vorticity eq. Scale analysis shows us that the solenoid term is small anyway.

4.4.3 Scale Analysis of Cartesian coordinate vorticity equation

$$\zeta = \left.\frac{\partial v}{\partial x}\right|_z - \left.\frac{\partial u}{\partial y}\right|_z \lesssim \frac{U}{L} \sim 10^{-5} s^{-1}$$

The $\lesssim$ symbol indicates approximately or less than because of partial cancellation. The ratio

$$\frac{\zeta}{f} \lesssim \frac{U}{f_o L} \equiv R_o \sim 10^{-1}$$

This is the Rossby number again. Recall it is also the ratio of the horizontal acceleration to the Coriolis Force. Hence, the vorticity dilution term can be approximated

$$(\zeta + f)\nabla \cdot \mathbf{V} \sim f\nabla \cdot \mathbf{V}$$

Holton shows that the solenoid term, horizontal vortex tilting term, and vertical advection of absolute vorticity are all small compared to vorticity dilution

Hence for normal synoptic scales

$$\frac{D_h}{Dt}(\zeta + f) \approx -f\nabla \cdot \mathbf{V},$$

or for particularly strong vorticity (a safe bet in any case)

$$\frac{D_h}{Dt}(\zeta + f) \approx -(\zeta + f)\nabla \cdot \mathbf{V}. \tag{1}$$

where the little h on the D stands for horizontal advection only

Two interesting points follow:

1) A consistency argument for estimating the scale of horizontal divergence.

- Recall $\nabla \cdot \mathbf{V}_g = 0$ for f=constant, so $\nabla \cdot \mathbf{V} \approx V_a/L \sim 10^{-6}\ s^{-1}$ where we have assumed $V_a \sim 1$ m/s is the characteristic scale of the ageostrophic wind.

- We can compute the scale of divergence from scale analysis of the vorticity eq.

  $$\frac{D_h}{Dt}(\zeta + f) \sim \frac{\partial \zeta}{\partial t} \text{ or } \mathbf{V} \cdot \nabla\zeta \lesssim \frac{U^2}{L^2} \sim 10^{-10}s^{-2}$$

  With $f \sim 10^{-4}s^{-1}$, we can solve for the divergence from Eq 1

  $$\nabla \cdot \mathbf{V} \lesssim 10^{-6}\ s^{-1}$$

2) An explanation for the relative strength of highs and lows

If we ignore advection for a moment so

$$\frac{D_h}{Dt}(\zeta + f) \to \frac{\partial \zeta}{\partial t}$$

then

$$\frac{\partial \zeta}{\partial t} \approx -(\zeta + f)\nabla \cdot \mathbf{V}.$$

- In the NH for cyclonic flow, both ζ and f are positive. Converging wind ($-\nabla \cdot \mathbf{V} > 0$ with $\zeta > 0$) causes $\partial\zeta/\partial t > 0$, so the vorticity increases further.

- In the NH for anticyclonic flow $\zeta < 0$ and $f > 0$. But normally $\zeta << f$ so $\zeta + f$ is usually positive. We normally see $\zeta < 0$ with diverging wind ($\nabla \cdot \mathbf{V} > 0$). But cancelation between ζ and f makes the divergence not very effective at making ζ decrease further. In fact as ζ approaches $-f$, $\partial\zeta/\partial t \to 0$, no amount of divergence matters.

4.5 Vorticity in barotropic fluids

4.5.1 Barotropic (Rossby) Potential Vorticity Equation

You have already worked homework problems using Holton's Eq 4.13

$$\frac{\zeta + f}{h} = \text{Const.}$$

Back in Holton section 4.3 this equation was derived from the circulation theorem for the special case of an incompressible homogeneous atmosphere. Implicit in this equation is that ζ is a vertical average.

We arrive at this conservation law by starting from the vorticity equation, keeping just the leading term from scale analysis (Eq 1 on the previous page). The result is known as the "barotropic potential vorticity equation". The name "barotropic" is a little misleading because we must assume the atmosphere is incompressible in the continuity equation, so

$$\nabla \cdot \mathbf{V} = -\frac{\partial \mathrm{w}}{\partial z}.$$

Thus Eq (1) becomes

$$\frac{D_h}{Dt}(\zeta + f) = (\zeta + f)\left(\frac{\partial \mathrm{w}}{\partial z}\right)$$

From here, a small number of steps given in Holton results in

$$\frac{D_h}{Dt}\frac{(\zeta + f)}{h} = 0$$

This equation is sometimes called the "incompressible barotropic" vorticity equation or the Rossby potential vorticity equation. In general the fluid depth is $h(x, y, t)$ and the height of both the top and bottom of the fluid can also vary spatially and in time (ie neither has to be flat). *This equation indicates that the creation of vorticity by vertical stretching of vortex tubes balances the horizontal advection of absolute vorticity.*

Recall barotropic means $\rho = \rho(p)$ and one of the consequences is that the geostrophic wind can have no vertical shear. We also know that the geostrophic wind is divergentless, provided we take f equal to a constant, and hence there is no vertical motion for purely geostrophic flow. Thus the vertical average of ζ in a barotropic fluid can be approximated: $\zeta \approx \zeta_g = \hat{k} \cdot \nabla \times \mathbf{V_g}$. Holton has ζ_g in the barotropic potential vorticity equation, but you will see it just as often with plain ζ, as it was earlier in the chapter.

4.5.2 Barotropic Vorticity Equation and the Streamfunction

If a barotropic, incompressible fluid has a constant depth h, then the flow is purely horizontal, independent of depth, and $\nabla \cdot \mathbf{V} = 0$. Thus Eq (1) is

$$\frac{D_h(\zeta + f)}{Dt} = 0$$

Again Holton has ζ_g in his equation, which is the case if we take f equal to a constant in our equation for the momentum, but only with regard to computing ζ. Keep in mind that you may compute ζ in both barotropic vorticity and potential vorticity equations from the true wind or the geostrophic wind.

For divergentless horizontal wind, the flow can be represented by a streamfunction Ψ, such that

$$u = -\frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \Psi}{\partial x}$$

And as long as the horizontal wind is divergentless,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \equiv \nabla^2 \Psi$$

Thus we can use the vorticity equation to forecast the vorticity. Then we can invert the Laplacian operator to find Ψ, and we can determine the wind field as a byproduct, iff the wind field is divergentless. Even if $\mathbf{V}$ is not divergentless, this procedure allows you to compute the divergentless part of $\mathbf{V}$, which you might like to call $\mathbf{V}_\Psi$

Happily the geostrophic wind is always divergentless, provided we take f equal to a constant, thus

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = f_o^{-1}\Big(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\Big) \equiv f_o^{-1}\nabla^2\Phi$$

In other words, Φ is a streamfunction of the geostrophic wind, provided we take f equal to a constant.

Recall that derivatives tend to yield noisy fields and integration tends to make things smoother. Thus it is natural to expect Ψ to be smooth (though wavy), while ζ often has much smaller scale features.