Chapter 6 Synoptic Scale Motions I: Quasi-Geostrophic Analysis

"The motion of large-scale atmospheric disturbances is governed by the laws of conservation of potential temperature and absolute potential vorticity, and by the conditions that the horizontal velocity be quasi-geostrophic and the pressure quasi-hydrostatic"
Jule Charney, 1948, from his paper titled *The scale of atmospheric motions*

6.2 QG Approximation

Goal: Develop scaled, conservative, time-dependent equations for baroclinic flow (ie, to predict $\mathbf{V}$ and T and to subsequently diagnose w to leading order). Motivation:

- Balanced equations like geostrophic wind or gradient wind are not predictive and pressure features are rarely linear or circular
- The real atmosphere is baroclinic. Recall barotropic means $\rho = \rho(p)$ only, so $\nabla_h T = 0$ and $\mathbf{V_g}$ is independent of height, which is unrealistic.

Begin with conservation equations in isobaric coordinates

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times \mathbf{V} = -\nabla\Phi \tag{1}$$

$$\frac{\partial\Phi}{\partial p} = -\alpha \tag{2}$$

$$\nabla \cdot \mathbf{V} + \frac{\partial\omega}{\partial p} = 0 \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) T - S_p\omega = J/c_p \tag{4}$$

with $S_p = -T\partial \ln\theta/\partial p$

The approximations in these equations:

- Eq 2 is the hydrostatic balance, which neglects vertical acceleration. It "filters" (or removes) vertical sound waves from the motion.

- curvature terms

The **QG approximation** eliminates more terms from scale analysis. Notably, it uses geostrophic balance in places, but without neglecting acceleration (otherwise we couldn't make a prediction). Thus the QG approx filters out even more unwanted motions (like hydrostaticity filters sound waves) that obstruct prediction!

Let $\mathbf{V} = \mathbf{V_g} + \mathbf{V_a}$ where $\mathbf{V_g}$ is nondivergent:

$$\mathbf{V_g} \equiv f_o^{-1}\mathbf{k} \times \nabla\Phi$$

This is the "CF" definition, where *the geostrophic wind is nondivergent because f_o is a constant.* Recall in a homework problem you showed $\mathbf{V_g}$ is nodivergent provided f is constant. Well now we redefine $\mathbf{V_g}$ replacing f with f_o explicitly for the QG approximation.

Recall if $\mathbf{V} = \mathbf{V_a} + \mathbf{V_g}$, then $|\mathbf{V_a}|/|\mathbf{V_g}| \sim \mathrm{R_o}$

Thus scale analysis shows that

$$\frac{D\mathbf{V}}{Dt} \approx \frac{D_g\mathbf{V_g}}{Dt}$$

where vertical advection is neglected and only the geostrophic wind participates in horizontal advection:

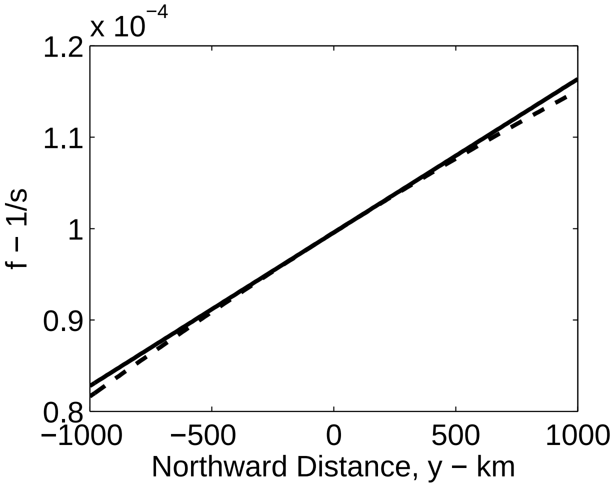
$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{V_g} \cdot \nabla.$$

The other two terms in the momentum equation are the Coriolis force and the PGF. We would not have a prognostic equation if these two terms are in balance — a zeroth order of magnitude approximation. Instead we must keep terms up to $\mathrm{R_o} \times |\nabla\Phi|$.

We do this by first making a Taylor's series expansion of f about latitude ϕ_o:

$$f \approx f_o + \beta y$$

where $\beta = (df/dy)_{\phi_o} = 2\Omega \cos\phi_o/a$ (a is the radius of Earth) and $y = a(\phi - \phi_o)$. This is called the **midlatitude beta-plane approximation** because f varies as if on a tangent plane rather than on the true curving earth. $\beta = 1.67 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$ at $\phi = 43°N$.



Hence,

$$f\mathbf{k}\times\mathbf{V} + \nabla\Phi = (f_o + \beta y)\mathbf{k}\times(\mathbf{V_g} + \mathbf{V_a}) - f_o\mathbf{k}\times\mathbf{V_g}$$

$$\approx f_o\mathbf{k}\times\mathbf{V_a} + \beta y\mathbf{k}\times\mathbf{V_g}$$

So Eq (1) becomes

$$\frac{D_g\mathbf{V_g}}{Dt} = -f_o\mathbf{k}\times\mathbf{V_a} - \beta y\mathbf{k}\times\mathbf{V_g} \tag{5}$$

Similarly the other conservations equations can make use of the definition of $\mathbf{V_g}$ and scale analysis. The TDE uses an expansion in temperature about a basic state temperature field $T_o(z)$ and a perturbation temperature field $T(x, y, z, t)$ plus hydrostaticity and the ideal gas law. Finally we arrive at the QG conservation equations:

$$\frac{D_g\mathbf{V_g}}{Dt} = -f_o\mathbf{k}\times\mathbf{V_a} - \beta y\mathbf{k}\times\mathbf{V_g} \tag{6}$$

$$\frac{\partial\Phi}{\partial p} = -\alpha = -\frac{RT}{p} \tag{7}$$

$$\nabla\cdot\mathbf{V_a} + \frac{\partial\omega}{\partial p} = 0 \tag{8}$$

$$\frac{D_g}{Dt}\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{R}{c_p}\frac{J}{p} \tag{9}$$

where the stability parameter is

$$\sigma = -\frac{RT_o}{p}\frac{d\ln\theta_o}{dp}.$$

Φ appears in the TDE rather than T to more easily relate to vorticity.

Recap:

- Hydrostatic equation (filters sound waves)
- QG Approximation (filters gravity waves) neglect $\mathrm{O}(\mathrm{R_o}^2 \times |\nabla\Phi|)$, e.g.,

$$\frac{D\mathbf{V}}{Dt} \approx \frac{D_g \mathbf{V_g}}{Dt}$$

- Midlatitude Beta plane approximation $f \approx f_o + \beta y$

6.2.2 QG Vorticity Eq

Goal: Derive QG Vorticity Eq and use it to predict Φ

With the CF definition of $\mathbf{V_g}$ so $\nabla \cdot \mathbf{V_g} = 0$, Φ is the streamfunction of the geostrophic wind:

$$u_g = -\frac{1}{f_o}\frac{\partial \Phi}{\partial y} \quad v_g = \frac{1}{f_o}\frac{\partial \Phi}{\partial x} \quad \text{and} \quad \zeta_g = \frac{1}{f_o}\nabla^2\Phi$$

QG Vorticity Eq:

$$\frac{\partial}{\partial x}[\mathrm{v-component\ of\ the\ QG\ Momentum\ Eq}]$$

$$\text{minus} \quad \frac{\partial}{\partial y}[\mathrm{u-component\ of\ the\ QG\ Momentum\ Eq}]$$

gives

$$\frac{D_g \zeta_g}{Dt} = -f_o \nabla \cdot \mathbf{V_a} - \beta v_g$$

or

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V_g} \cdot \nabla \zeta_\mathbf{g} - \beta v_g - f_o \nabla \cdot \mathbf{V_a}$$

which is like Eq 4.22, $D_h(\zeta + f)/Dt = -f\nabla \cdot \mathbf{V}$, but with the important QG approx to filter gravity waves.

For disturbances embedded in the midlatitude westerlies the first two terms on the r.h.s. have opposite sign. See Holton fig 6.7. If the advection of relative vorticity dominates, it tends to cause the pattern of highs and lows to shift eastward, as if the systems were on a conveyer belt. If instead the advection of planetary vorticity dominates it tends to make the pattern shift westward, or *retrogress.*

The dominance of one term over the other depends on scale. Synoptic scale systems tend to advect westward, while planetary scale systems should retrogess according to QG theory and without taking into account topography, land/sea heating, etc.

The dependence on scale is illustrated nicely by an example of a sinusoidal wave disturbance embedded in westerly flow:

$$\Phi(x, y) = \Phi_0 - f_o U y + f_o (V/k) \sin kx \cos ly. \tag{10}$$

The wave numbers k and l are $k = 2\pi/L_x$ and $l = 2\pi/L_y$ with L_x and L_y the wavelength in the x and y directions.

In class we worked out how

$$u_g = -\frac{1}{f_o}\frac{\partial \Psi}{\partial y} = U + u_g' = U + V\frac{l}{k}\sin kx \sin ly$$

$$v_g = \frac{1}{f_o}\frac{\partial \Psi}{\partial x} = v_g' = V \cos kx \cos ly$$

$$\zeta_g = \frac{1}{f_o}\nabla^2\Phi = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = -V\frac{(k^2 + l^2)}{k}\sin kx \cos ly$$

Advection of relative vorticity is

$$-\mathbf{V_g}\cdot\nabla\zeta_g = -(u_g\hat{\imath} + v_g\hat{\jmath})\cdot\left(\frac{\partial\zeta_g}{\partial x}\hat{\imath} + \frac{\partial\zeta_g}{\partial y}\hat{\jmath}\right) = -u_g\frac{\partial\zeta_g}{\partial x} - v_g\frac{\partial\zeta_g}{\partial y}$$

For the special case of a sinusoidal wave disturbance (not gaussian or some other non wave disturbance), the wave-wave contribution to the advection of relative vorticity is zero. Hence only the zonal mean wind advects the vorticity:

$$-\mathbf{V_g}\cdot\nabla\zeta_g = -U\frac{\partial\zeta_g}{\partial x} = UV(k^2 + l^2)\cos kx \cos ly$$

Where there is a trough to the west and ridge to east embedded in westerly mean wind

$\zeta_g > 0$ $\quad\rightarrow U\quad$ $\zeta_g < 0$

At the midpoint between the trough and ridge you should expect advection to be positive.

You can see this by looking upstream and seeing higher voriticy than at the midpoint — the flow is bringing the trough towards you!

Advection of planetary vorticity is

$$-\beta v_g = -\beta V \cos kx \cos ly$$

The ratio of the advection of relative vorticity to advection of planetary vorticity is

$$\frac{U}{\beta}(k^2 + l^2) = \frac{4\pi^2 U}{\beta}\left(\frac{1}{L_x^2} + \frac{1}{L_y^2}\right)$$

Therefore the advection of relative vorticity dominates provided the scale of the wave is synoptic $L_x \sim L_y \sim 3000$ km, while the advection of planetary vorticity dominates for $L_x \sim L_y \sim 10,000$ km.

At this point we can make our Φ time dependent by only changing $x \rightarrow x - ct$ to all the equations on the previous page, so the whole system is moving eastward with phase speed c. Once a system like this is set in motion and if the wave disturbance is height dependent so there is "differential vorticity advection" (ie the magnitude of the disturbance or its phase changes with height, like in problem 6.4), then the system will develop vertical shear of the horizontal wind. This shear drives an ageostrophic vertical motion via the stretching term, $\partial\omega/\partial p$ in

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V_g} \cdot \nabla \zeta_{\mathbf{g}} - \beta v_g + f_o \frac{\partial \omega}{\partial p}$$

To compute $\partial\omega/\partial p$, we must first find

$$\frac{\partial \zeta_g}{\partial t} = Vc(k^2 + l^2)\cos(k(x - ct)\cos ly$$

Hence

$$\frac{\partial \omega}{\partial p} = \frac{1}{f_o}\left(\frac{\partial \zeta_g}{\partial t} + \mathbf{V_g} \cdot \nabla \zeta + \beta v_g\right) = -\frac{V}{f_o}[(U - c)(k^2 + l^2) - \beta]\cos(k(x - ct)\cos ly$$

and of course the streching $\partial\omega/\partial p = -\nabla \cdot \mathbf{V_a}$

Unlike advection of relative vorticity, there is no simple rule to determine where there is divergence with respect to the vorticity. Instead we must solve the vorticity equation and determine $\partial\omega/\partial p$ from the balance of terms. For the example given here the relative magnitudes of U, c, and β are key to determining the sign of the divergence.

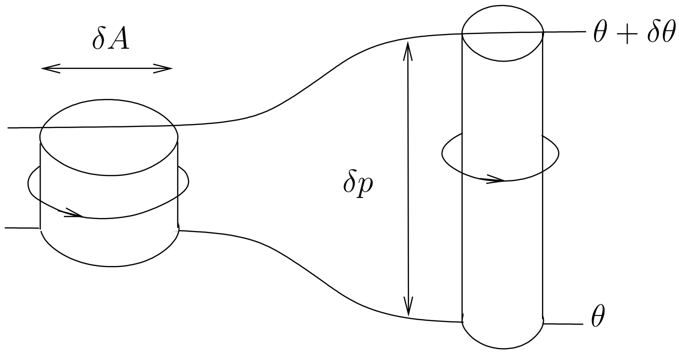
6.3 QG Prediction

From the QG vorticity equation, we know that given some positive vertical stretching, the QG vorticity will rise. The vertical stretching will necessarily coincide with adiabatic heating, which then influences the temperature as can be seen by taking the derivative of the TDE with respect to p:

$$\frac{\partial}{\partial p}\left[\frac{fo}{\sigma}\left(\frac{\partial}{\partial t}+\mathbf{V_g}\cdot\nabla\right)\left(-\frac{\partial\Phi}{\partial p}\right)\right]-f_o\frac{\partial\omega}{\partial p}=f_o\frac{\partial}{\partial p}\left[\frac{R}{\sigma c_p}\frac{J}{p}\right]$$

Alternatively when the temperature changes following horizontal parcel motion varies with height (as one expects), then the first term above is nonzero. This drives an ageostrophic vertical motion which yields vertical streching and drives vorticity changes. There is a strong relation between the QG vorticity equation and the TDE that is a result of the conservation laws.

Remember this figure? It illustrates vortex stretching, $\partial\omega/\partial p > 0$ as the parcel moves from left to right. The upper part of the vortex moves up, so it cools, and the lower part moves down, so it warms. This drives a change in the parcel's vertical temperature profile. Thus we can associate such changes in the vertical temperature profile with vortex stretching (and vice versa).



6.3.1 Geopotential Tendency Equation

A consequence of vertical streching appearing in both the QG vorticity equation and the equation above is that we can combine the equations and eliminate the vertical stretching:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_o^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{\text{A}} = \underbrace{-f_o\mathbf{V_g}\cdot\nabla\left(\frac{1}{f_o}\nabla^2\Phi + f\right)}_{\text{B}}\underbrace{-\frac{\partial}{\partial p}\left[-\frac{f_o^2}{\sigma}\mathbf{V_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\text{C}}$$

where "chi" $\chi = \partial\Phi/\partial t$ and $J = 0$. If we know Φ at one time, then we can compute its tendency, χ.

Term A is the local geopotential tencency, which when Φ is composed of a wave disturbance tends to vary like minus one times χ (recall the second derivative of a cosine is minus cosine). It is a combination of the time rate of change of the QG vorticity and the change to the vertical temperature profile that arises from vortex stretching from the TDE (this temperature change is due to the ageostrophic motion!).

Term B is proportional to the advection of absolute vorticity from the QG vorticity equation

Term C is the differential temperature (or thickness) advection. This term causes the upper level disturbance to develop. For example, as illustrated in Fig 6.5 when below and to the east of the 500 hPa trough, there is cold advection associated with the cold front. This will cause the trough to deepen and move to the east. The temperature change that results from term C is due to the geostrophic motion.

6.3.2 Quasi-Geostrophic Potential Vorticity

An equivalent form of the QG geopotential tendency equation is

$$\frac{D_g q}{Dt} = 0$$

$$q = \frac{1}{f_o}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_o}{\sigma}\frac{\partial\Phi}{\partial p}\right)$$

which states that the sum of the QG relative vorticity, the planetary vorticity, and the stretching vorticity is conserved following the geostrophic motion.

Recall the Ertel's potential vorticity

$$P = (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

is conserved following the adiabatic parcel motion. q is a linearized form of P.

Important properties:

- The laplacian of a wave function is proportional to minus the same wave function. Hence, according to the first term in q, when q increases, you can expect trough development and vice versa.
- When $\mathbf{V_g}$ is parallel to lines of constant q, the advection of q is zero, so the tendency of q must also be zero and hence the flow must be steady.

6.3.3 - 6.3.4 PV inversion and vertical coupling

Because q is composed in part of second order derivatives of Φ, q tends to have smaller-scale variations (it can even be discontinuous), while Φ tends to be more smoothly varying and spread out.

For Φ of the form from p5 of last week's notes:

$$\Phi(x, y) = \Phi_0 - f_o U y + f_o(V/k)\sin kx \cos ly$$

If we let σ and f be constants for simplicity here, then

$$q = \frac{1}{f_o}\nabla^2\Phi + f_o + \frac{f_o}{\sigma}\frac{\partial^2\Phi}{\partial p^2}$$

and $q = f_o + Q(p)\sin kx \cos ly$, where $Q = -(k^2+l^2)V/k + (f_o/\sigma/k)\partial^2 V/\partial p^2$. The two terms in the conservation of q are

$$-\mathbf{V_g}\cdot\nabla q = -U\frac{\partial q}{\partial x} = -kUQ\cos kx \cos ly$$

and

$$\frac{\partial q}{\partial t} = \frac{1}{f_o}\nabla^2\chi + \frac{f_o}{\sigma}\frac{\partial^2\chi}{\partial p^2}$$

where order of the spatial derivative and time derivatives have been interchanged. Equating these last two:

$$\frac{1}{f_o}\nabla^2\chi + \frac{f_o}{\sigma}\frac{\partial^2\chi}{\partial p^2} = -kUQ\cos kx\cos ly$$

and substituting

$$\chi(x, y, p, t) = X(p, t)\cos kx\cos ly$$

yields

$$\frac{d^2X}{dp^2} - \lambda^2 X = -\frac{\sigma}{f_o}kUQ$$

where $\lambda^2 = (k^2 + l^2)\sigma f_o^{-2}$. This is a diffusion equation with a source term. It indicates that QG PV advection at a given altitude will create a Φ spread out in p with vertical scale λ^{-1}. Because λ depends inversely on the horizontal length scale, the larger the length scale, the more spread out in the vertical is the response. Hence the advection of q at upper-levels from very large length-scale disturbances and will tend to induce Φ tendencies down to the surface with little loss of amplitude.

6.4 Diagnosis of the vertical motion

6.4.1 The traditional omega equation

We have been computing ω in problems where we made use of a tendency (time derivative) of another variable. For example when we used the QG vorticity equation, we had the tendency of ζ_g. Atmospheric measurements are not routinely conducted frequently enough to accurately capture the tendency of variables, so it would be better to find ω from an equation without any time derivatives. This can be accomplished by performing operations on the vorticity equation and the TDE, so that there sum eliminates the tendency terms. The result is the traditional omega equation:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_o^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\omega}_{\text{A}} = \underbrace{\frac{f_o}{\sigma}\frac{\partial}{\partial p}\mathbf{V_g}\cdot\nabla\left(\frac{1}{f_o}\nabla^2\Phi + f\right)}_{\text{B}} + \underbrace{\frac{1}{\sigma}\nabla^2\left[\mathbf{V_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\text{C}} - \underbrace{\frac{\kappa}{\sigma p}\nabla^2 J}_{\text{D}}$$

Term A tends to be proportional to minus ω and recall that ω is proportional to minus w, therefore term A tends to be proportional to w.

Term B is the rate of change of the absolute vorticity advection with height

Term C is the horizontal Laplacian of temperature advection

Term D is the horizontal Laplacian of diabatic heating

Note that terms B-D depend on σ^{-1} therefore they decrease with increasing stability. In other words, vertical motion is weak in a highly stable atmosphere.

There is often some cancellation between terms B and C and individually these terms are not invariant under a Galilean transformation (ie $x \rightarrow x - ct$), hence after some manipulation, neglecting small terms, and letting $J = 0$ we can also write:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_o^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\omega}_{\text{A}} \approx \frac{f_o}{\sigma}\frac{\partial \mathbf{V_g}}{\partial p}\cdot\nabla\left(\frac{2}{f_o}\nabla^2\Phi + f\right)$$

Note that my equation differs from Holton's Eq 6.36 by a factor of 2 on the right. I performed a scale analysis on the term from the traditional omega equation that Holton neglected and I didn't find it negligible. I read a couple more texts about the omega equation and they agreed with me. This factor of two gives you the correct answer in problem 6.8 too. Fortunately the conclusions Holton draws in the text are the same, since they do not depend on the factor of 2!

The advection on the right is accomplished by the thermal wind, not simply the geostrophic wind because

$$\frac{\partial \mathbf{V_g}}{\partial p} \approx \frac{\mathbf{V_g}(p_o) - \mathbf{V_g}(p_1)}{\Delta p} = -\frac{\mathbf{V_T}}{\Delta p}.$$

Thus

$$\text{w} \propto -\mathbf{V_T}\cdot\nabla(\zeta_g + f)\begin{cases} < 0 & \text{east of the 500-hPa ridge} \\ > 0 & \text{east of the 500-hPa trough} \end{cases}$$

Finally, for synoptic scale disturbances, the relative vorticity advection dominates, and for planetary waves the planetary vorticity advection dominates.