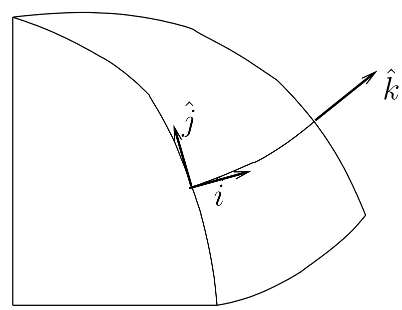
Chapter 1 Notes

A Note About Coordinates

We nearly always use a coordinate system in this class where the vertical, $\hat{k}$, is normal to the Earth's surface and the x-direction, $\hat{\imath}$, points to the east and the y-direction, $\hat{\jmath}$, points to the north.



1.4.1 Pressure Gradient Force

Consider an air parcel/point in a field/volume of air $\delta V = \delta x \delta y \delta z$:

On side wall labeled A: we can express the pressure in a Taylor's series expansion:

$$p_A = p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \text{higher order terms},$$

where p_0 is the pressure at the middle of the volume.

The pressure force acting on the volume at A is then

$$F_{Ax} = -p_A \delta y \delta z = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$$

Likewise at B but the direction is opposite (note carefully all signs here):

$$F_{Bx} = p_B \delta y \delta z = \left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y \delta z$$

The net force in the x direction is:

$$F_x = F_{Ax} + F_{Bx} = -\frac{\partial p}{\partial x}\delta x \delta y \delta z$$

The mass of the parcel is $m = \rho\delta x\delta y\delta z$, hence

$$\frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

And in 3D

$$\frac{\mathbf{F}}{m} = -\frac{1}{\rho}\nabla p$$

1.4.2 Gravitational Force

$\mathbf{g}^*$ is the gravitational force. It always points towards the center of mass, and it is proportional to the inverse square of the distance above the center of mass:

$$\mathbf{g}^*(\mathbf{z}) = \frac{-GM}{(a+z)^2}\hat{r},$$

where a is Earth's radius, z is the height above the surface, G is the gravitational constant, M is Earth's mass, and $\hat{r}$ is a unit vector directed from the center of mass to the point where the force is evaluated. Hence, the magnitude

$$g^*(z) = \frac{GM}{(a+z)^2} = \frac{GM}{a^2}\frac{1}{(1+z/a)^2} = g_0^*\frac{1}{(1+z/a)^2}$$

where g_0^* is the gravitational force at the surface. For the troposphere $z \sim$10 km and the Earth approximate radius is $a \sim 10^4$ km, so

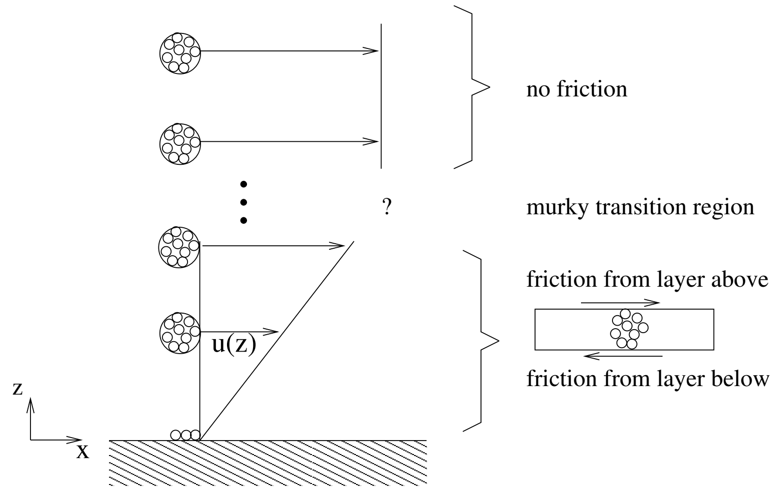
$$g^*(z) \sim g_0^*\frac{1}{(1+10\text{ km}/10^4\text{ km})^2}$$

and we can safely let $g^*(z) = g_0^*$.

1.4.3 Viscous or Internal Frictional Force

Here I present the same concept as section 1.4.3 in Holton, but using a more geophysical perspective. At this time we consider molecular viscosity alone. Viscosity due to turbulent eddies, which is also a source of friction in the atmosphere, is covered in Chapter 5,

Consider a steady laminar (ie non turbulent) flow above the ground as shown in the figure. Flow near the ground is analogous to flow in between Holton's pair of plates in Fig 1.3, but seems less complicated.



Internal friction resists flow wherever there are flow gradients, and surface friction prevents any flow of air parcels in contact with the ground. Usually there is a layer near the ground with strong flow gradients. For flow that increases linearly with height, as shown in the figure, the flow is at steady state. Within this "frictional layer", friction at the upper and lower edges of a fluid layer is in balance. At some height though there must be other force(s) besides friction, such as a pressure gradient force, that results in flow to the right. Holton accomplishes this with a moving horizontal plate.

Within the frictional layer, the tangential force per unit area, or "stress", from friction is written heuristically as

$$\tau = \mu \frac{u(z)}{z}, \tag{1}$$

where μ is the *dynamic coefficient of viscosity.*

Although air parcels are moving to the right, random molecular motions take place in all directions. When slower moving molecules from below change places (or collide) with more rapidly moving molecules aloft, there is a net downward transport of x momentum, even at steady-state. This momentum transport is the shearing stress. It exterts a drag force on the surface.

For non-steady flow, u may not increase linearly with height. Hence, in the x-direction, the vertical stress component of a parcel depends on the wind shear:

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}. \tag{2}$$

Acting on a fluid element,
pressure is a normal force per unit area and
stress is a tangential force per unit area,

so
a pressure gradient yields a net force on the fluid element and
a stress gradient yields a net force on the fluid element.
Hence, we can recycle the derivation for the net force a parcel experiences in the presence of a stress gradient as we did for a pressure gradient here. Compare Holton figures 1.4 and 1.1 and you should see that the force per unit mass is

$$\frac{F_{zx}}{m} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{3}$$

Now substitution from Eq (2) gives

$$\frac{F_{zx}}{m} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}. \tag{4}$$

Of course we have only considered the z-direction shear of the u-wind. There are actually nine possible combination of subscripts in total. Holton writes out all the possible combinations in his Eq 1.5, and drops the mass m from previous sections by redefining F to be the force per unit mass. We can rewrite Holton's Eq 1.5 in compact vector notation as

$$\mathbf{F_r} = \nu \nabla^2 \mathbf{U}, \tag{5}$$

where $\nu = \mu/\rho$. Vertical wind shear is usually much larger than horizontal wind shear in the atmosphere, so we can usually get away with just computing

$$\mathbf{F_r} = \nu \frac{\partial^2 \mathbf{V}}{\partial z^2}. \tag{6}$$

Note upper case $\mathbf{U}$ is the 3D velocity vector while upper case $\mathbf{V}$ is only in the horizontal plane.

The relative importance of friction can be estimated from the Reynold's number:

$$Re = \left|\frac{D\mathbf{V}}{Dt}\right| \bigg/ \left|\nu \frac{\partial^2 \mathbf{v}}{\partial z^2}\right|. \tag{7}$$

where $\frac{D\mathbf{V}}{Dt}$ is acceleration, which is typically about 10^{-3} m s^{-1}. Soon you will compute the order of magnitude for friction of typical atmospheric flows.

1.5 Non inertial Reference Frames and "Apparent Forces"

1.5.1 Centripetal Acceleration/ Centrifugal Force

Centripetal acceleration is

$$\frac{D\mathbf{V}}{Dt} = -\omega^2 r \hat{r}.$$

(See Holton for a derivation)

Capital-D derivatives are material derivatives, which are derivatives taken along a path following the motion. The math is no different from small-d derivatives, which are sometimes called total derivatives. Total derivatives are more general for example you may be interested in df/dt for f(T(t),P(t)):

$$\frac{df}{dt} = \frac{\partial f}{\partial T}\frac{dT}{dt} + \frac{\partial f}{\partial P}\frac{dP}{dt}.$$

In fact, you have dealt with capital D derivatives for a long time, but nobody mentioned it. Acceleration is always a material derivative.

A quick review of derivatives

d/dt and D/Dt are both "total derivatives". There is no difference between capital and lowercase d-derivatives. From the chain rule if $w = f(x)$ and $x = g(t)$ then

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

$\partial/\partial t$ is a "partial derivative", used to take derivatives of a multi-variate function with respect to one variable Again the chain rule if $w = f(x, y)$, $x = g(t)$ and $y = h(t)$ then

$$\frac{dw}{dt} = \left.\frac{\partial w}{\partial x}\right|_y \frac{dx}{dt} + \left.\frac{\partial w}{\partial y}\right|_x \frac{dy}{dt}.$$

The vertical bar indicates what is held fixed while taking the partial derivatives, though the vertical bars are usually not written explicitly.

The "exact differential" of $w = f(x, y)$ is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

where we consider both ways dw can vary with respect to its dependent variables.

1.5.2 Gravitational Force

An object <u>at rest</u> on Earth but not at the poles is in an accelerating reference frame — centripetal acceleration that is. Its apparent force is

$$\mathbf{F_{Ce}} = \Omega^2 R\hat{R}$$

where $\mathbf{R}$ is the vector from Earth's axis to the object.

Imagine a perfectly spherical and frictionless Earth. Because of $\mathbf{F_{Ce}}$, objects at rest on this Earth would slip towards the equator. Earth is made of rocks that cannot sustain a shear force, so rocks would slip towards the equator on a spherical Earth too! Rotation has therefore caused Earth to change shape so that the tangential component of $\mathbf{g}^*$ at the surface exactly balances the tangential component of $\mathbf{F_{Ce}}$. This is very fortunate for students of atmospheric sciences!

$\mathbf{g}^*$	true gravity	Always points towards the center of mass so it not everywhere normal to Earth's surface. It also varies with latitude on the surface.
$\mathbf{g} = \mathbf{g}^* + \Omega^2 R\hat{R}$	convenient gravity	normal to the surface. It varies with latitude on the surface

Now imagine Earth is the shape it is, still frictionless, but not rotating. Put an object at the north pole (NP) and give it a shove southward. Because the Earth is not a sphere, $\mathbf{g}^*$ has a component that is tangent to the surface, and pulls the object back towards the NP. Because the Earth is frictionless, it overshoots and we get simple harmonic motion:

$$\frac{DR}{Dt} + \Omega^2 R = 0.$$

It actually doesn't matter if the Earth were rotating, we would still get the same motion from the fixed reference frame because the Earth merely slides around underneath the object, which doesn't notice. Formally this is because the object has no angular momentum at the start and the frictionless Earth imparts no torque. From the rotating reference frame, this object appears to make circles every 12 hours. This is almost impossible to visualize from Fig 1.7, so I'll put some animations on the web to help.

It is convenient to define the "geopotential" function ϕ: $\nabla\phi = -\mathbf{g}$. Because $\mathbf{g} = -g\hat{k}$, $\phi = \phi(z)$ and $d\phi/dz = g$.

1.5.3 The Coriolis Force and Curvature Effect

It is possible and perhaps even simpler to derive Holton's Eq 1.10a and b by setting the time derivative of the total angular momentum component along Earth's axis equal to zero:

$$\frac{D}{Dt}\left[R(\Omega R + u)\right] = 0, \tag{8}$$

which can be written

$$2\Omega R\frac{DR}{Dt} + u\frac{DR}{Dt} + R\frac{Du}{Dt} = 0 \tag{9}$$

or

$$\frac{Du}{Dt} = -\frac{(u + 2\Omega R)}{R}\frac{DR}{Dt}. \tag{10}$$

With $R = (a + z)\cos\phi$,

$$\frac{DR}{Dt} = \cos\phi\frac{Dz}{Dt} - (a + z)\sin\phi\frac{D\phi}{Dt}.$$

Making the substitutions $Dz/Dt = w$ and $aD\phi/Dt = v$ and recognizing that $z << a$, gives

$$\frac{Du}{Dt} \approx -2w\Omega\cos\phi + 2v\Omega\sin\phi + \frac{uv\tan\phi}{a} - \frac{uw}{a}. \tag{11}$$

This equation is a combination of Holton's Eq 1.10a and b for the general case of a displacement (ie an impulse that imparts an initial velocity) in the meridional or vertical. The first two terms are the Coriolis force and the last two are the curvature effect, both for the $\hat{x}$ direction only.

The Coriolis Force and the curvature effect in $\hat{y}$ and $\hat{z}$ directions arise from the influence of east-west motions on the centrifugal force. The net force per unit mass in the $\vec{R}$ direction owing to velocity component u is

$$\vec{F}(u) = \left(\Omega + \frac{u}{R}\right)^2 \vec{R} - \Omega^2\vec{R} = \left(2u\Omega + \frac{u^2}{R}\right)\hat{R}. \tag{12}$$

With $\hat{R} = -\hat{y}\sin\phi + \hat{z}\cos\phi$ and approximating $R \approx a\cos\phi$ from the start*, the force can be broken into $\hat{y}$ and $\hat{z}$ components. If no other forces are acting, the acceleration is equal to this force alone (ie $F_y(u) = Dv/Dt$):

$$\frac{Dv}{Dt} = -2\Omega u\sin\phi - \frac{u^2}{a}\tan\phi \tag{13}$$

and

$$\frac{Dw}{Dt} = 2\Omega u\cos\phi - \frac{u^2}{a}. \tag{14}$$

*We cannot make the approximation $a+z \approx a$ right away in the derivation of Du/Dt because it is operated on by D/Dt, and $D(a + z)/Dt = Dz/Dt$.

For synoptic-scale motions $u << \Omega R$ so we can usually ignore the curvature terms (the terms with a in the denominator), and $w << u$ so the first term in Du/Dt is negligible. Normally the vertical component of the Coriolis force is much smaller than gravity, so it is ignored too. In summary then

$$\left(\frac{Du}{Dt}\right)_{Co} \approx 2\Omega v\sin\phi = fv \tag{15}$$

and

$$\left(\frac{Dv}{Dt}\right)_{Co} \approx -2\Omega u\sin\phi = fu. \tag{16}$$

One final note. Once an objects begins to move in a particular direction, the Coriolis force and curvature effects produce a force to the right of the motion in the northern hemisphere. For example, an initial nudge in the $\hat{x}$ direction will create departures in the $\hat{y}$ and $\hat{z}$ directions. Hence equations for Du/Dt, Dv/Dt, and Dw/Dt are coupled and generally may not be used in isolation (unlike the example in Holton on page 18).

1.6.1 Hydrostatic Equation

Hydrostatic Balance (for atmosphere at rest $p = p(z)$):

$$g = -\frac{1}{\rho}\frac{dp}{dz}$$

Later we will show that it is a good approximation for synoptic scale motions too, when $p = p(x, y, z, t)$:

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Substitution from the ideal gas law and integrating gives the hypsometric equation:

$$\Phi(z_2) - \Phi(z_1) = g_o(Z_2 - Z_1) = R\int_{p_2}^{p_1} T d\ln p$$

The average temperature defined for $\ln p$ vertical coordinate is

$$< T >= \int_{p_2}^{p_1} T d\ln p \left[\int_{p_2}^{p_1} d\ln p\right]^{-1} \tag{17}$$

can be used to define a "scale height" $H = R < T > /g_o$, which is typically about 8 km. Now the hypsometric equation can be written:

$$Z_T = Z_2 - Z_1 = H\ln(p_1/p_2) \tag{18}$$

or starting at $Z_1 = 0, p_1 = p_o, Z_2 = Z, p_2 = p$:

$$p(Z) = p_o e^{-Z/H}$$

Sample Problem (useful for working homework 2)

In an atmosphere with constant lapse rate

$$\frac{dT}{dz} = -\gamma$$

so $T = T_o - \gamma z$, where we have let $Z_1 = 0, Z_2 = Z$ and T_o be the temperature at the ground. For $p_1 = p_o, p_2 = p$, and assuming $g = g_o$ (so $z = Z$), what is $p(z)$?

First use hydrostic balance and the ideal gas law to write

$$\frac{dp}{dz} = -\frac{pg}{RT}$$

We must first be sure to have only p's, z's, and constants on the right, OR we must rewrite the derivative on the left to match the variables on the right, namely replace dz with dT somehow. One way will be worked in class according to democratic vote. You can fill in the other method on your own.

What is the difference between between $< T >$ (see Eq 1) and $\bar{T}$ where

$$\bar{T} = \frac{\int_{Z_1}^{Z_2} T dz}{(Z_2 - Z_1)}?$$

Obviously there is no difference if T is a constant, so let's show how the averages differ for a constant lapse rate (i.e., constant temperature gradient with height) atmosphere. First

$$\bar{T} = \frac{\int_0^Z (T_o - \gamma z) dz}{Z} = T_o - \gamma Z/2$$

We can get $< T >$ from Eq 17

$$< T > = -\frac{Zg}{R} \Big/ \ln(p/p_o)$$

which appears pretty different, although we shall see below these equations give numbers that are similar for typical conditions.

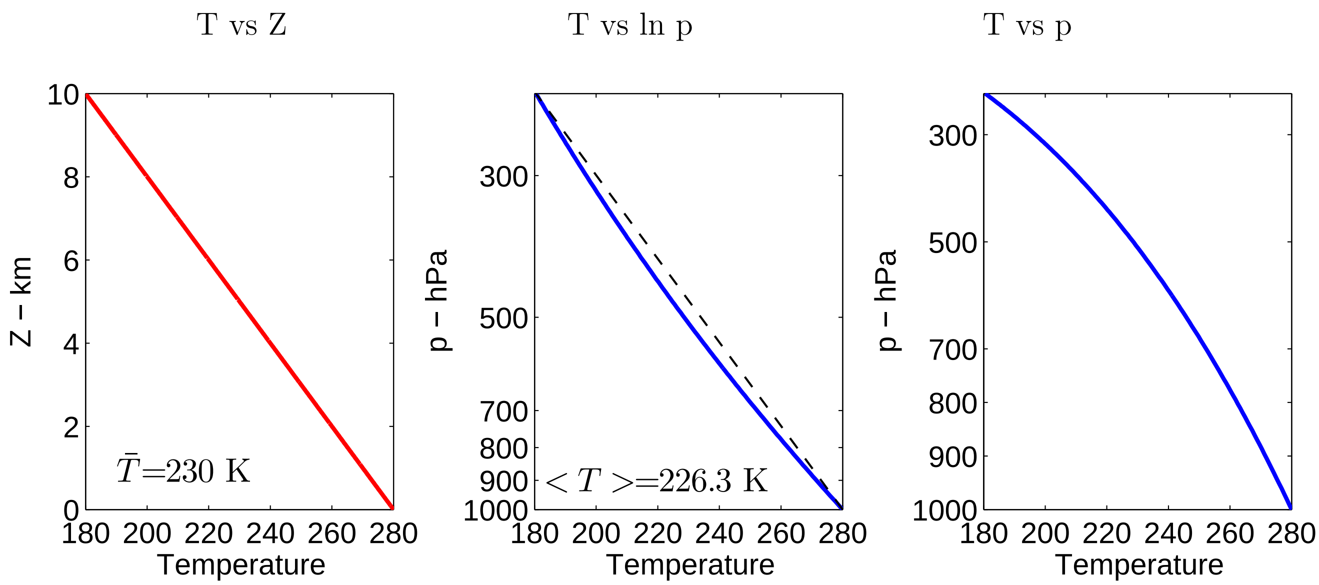
We find they differ, but why? We can recycle from the previous page:

$$\frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{R\gamma/g}$$

to find $T(\ln p)$:

$$T = T_o \exp\left[\frac{R\gamma}{g} \ln \frac{p}{p_o}\right]$$

which is exponential in ln p, while T(z) is linear in z. The exponent decreases with -ln p but the e-folding value is very large, so it appears fairly linear for reasonable values of γ. For the figure below, p_o=1000 hPa, Z=10 km, T_o = 280 K, and γ = 10K/km (ie the lapse rate for a dry atmosphere, a moist one has an even smaller γ and therefore $T(\ln p)$ is even more linear.)



1.6.2 Pressure as a vertical coordinate - isobaric coordinates

Often pressure is used as a vertical coordinate in the atmosphere When this is the case, we need to re-express the horizontal pressure gradient force in terms of a derivative of something at constant pressure instead. Hence for the x-component we want to write

$$\mathrm{PGF}_x = -\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_z \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial x}\right|_p$$

This is most easily done by considering the sketch:

and recognizing that

$$\left(\frac{\delta z}{\delta x}\right)_p = \left[\frac{(p_o + \delta p) - p_o}{\delta x}\right]_z \Bigg/ \left[\frac{(p_o + \delta p) - p_o}{\delta z}\right]_x$$

or

$$\left.\frac{\partial z}{\partial x}\right|_p = -\left.\frac{\partial p}{\partial x}\right|_z \Bigg/ \left.\frac{\partial p}{\partial z}\right|_x$$

Substitute from the hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g$ and rearranging gives

$$\mathrm{PGF}_x = -\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_z = -\left.\frac{\partial \Phi}{\partial x}\right|_p$$

So we can swap coordinates from $p(x, y, z, t)$ to $\Phi(x, y, p, t)$ in isobaric coordinates and happily no density appears in "PGF" on right