Jan 5, 2011

Numerical solutions to wave equations

(1) $$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

(2) $$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x}$$

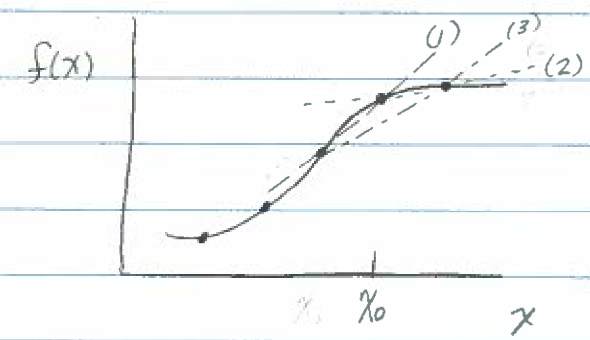
"one sided"

(3) $$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x}$$ "centered"

If $\frac{df}{dx}(x_0)$ is continuous and $\Delta x \to 0$

all three are identical

But if Δx is finite, (1) - (3) give different approximations to $\frac{df}{dx}$ at x_0



points are seperated by Δx

when Δx is finite (1) - (3) are called finite differences

e.g. $$\frac{df}{dx}(x_0) \approx \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

What is the accuracy of this approximation?

use Taylor's Series Expansion

$$f(x_0+\Delta x) = f(x_0) + \Delta x \frac{df}{dx}(x_0) + \frac{\Delta x^2}{2}\frac{d^2f}{dx^2}(x_0) + \cdots$$

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - \frac{df}{dx}(x_0) = \frac{\Delta x}{2}\frac{d^2f}{dx^2}(x_0) + \dots$$

one-sided differences
leading term is of order Δx or $\mathcal{O}(\Delta x)$
"first - order accurate"

$$\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} - \frac{df}{dx}(x_0) = \frac{\Delta x^2}{6}\frac{d^3f}{dx^3}(x_0) + \dots$$

Centered difference are $\mathcal{O}(\Delta x^2)$

Consider the constant wind speed advection eq

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$$

Solution $\psi(x,t) = \psi(x-ct, 0)$ "Conveyor belt"

We wish to obtain an approximate solution at grid point $(n\Delta t, j\Delta x)$ where n & j are integers

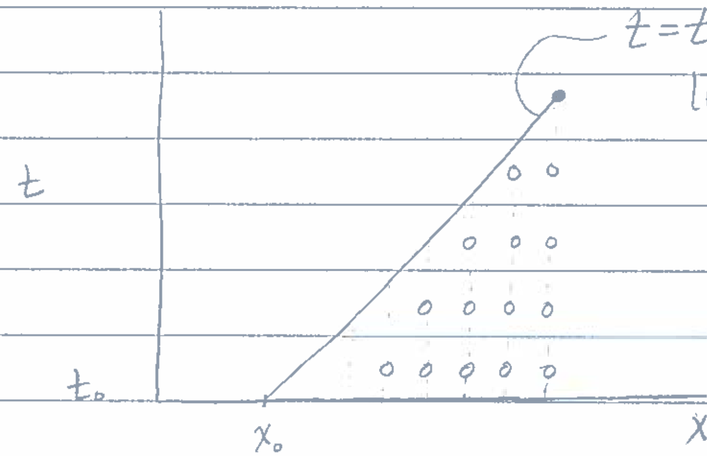
Using $\psi(n\Delta t, j\Delta t) = \psi_j^n$ shorthand

one possible solution is the upstream scheme

$$\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t} + c\,\frac{\psi_j^n-\psi_{j-1}^n}{\Delta x} = 0$$

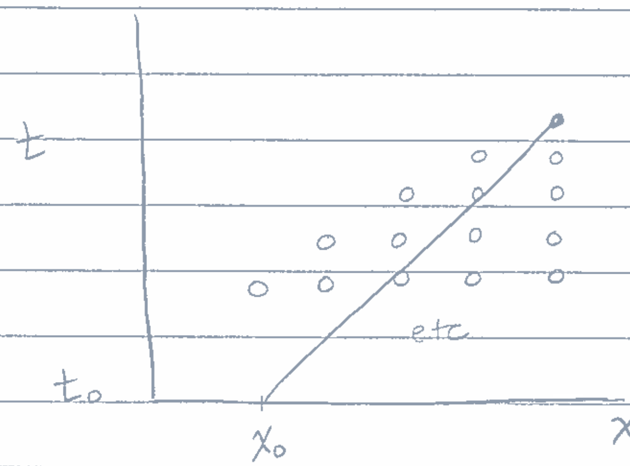
$$\psi_j^{n+1} = \psi_j^n - \frac{c\Delta t}{\Delta x}\left(\psi_j^n - \psi_{j-1}^n\right)$$

$\frac{c\Delta t}{\Delta x}$ = "Courant" Number



$t = t_0 + \frac{1}{c}(x - x_0)$
line known as "characteristic" is the true domain of influence of the point (x_0, t_0)

Circles are the domain of dependence of the upstream solution.
The circles must encompass the line if the solution is stable. This is not the case above but is below



What is the key difference?
Δx versus Δt

The Stability condition is: $\frac{\Delta t}{\Delta x} < \frac{1}{c}$

The Courant Fredrich Lewy Condition (CFL) for upstream

$$0 \le \frac{c\Delta t}{\Delta x} \le 1$$

(the zero is a lower limit because $c, \Delta t, \Delta x$ are each ≥ 0)

A few other solutions are

forward in time - centered in space

$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} + c\left(\frac{\psi_{j+1}^n - \psi_{j-1}^n}{2\Delta x}\right) = 0$$

Leapfrog in time - centered in space

$$\frac{\psi_j^{n+1} - \psi_j^{n-1}}{\Delta t} + c\left(\frac{\psi_{j+1}^n - \psi_{j-1}^n}{2\Delta x}\right) = 0$$

The different scheme have different behaviors that you will explore in HW1

B.C.'s

Initial Condition

$\psi_0 = \sin^6 kx$

B.C "periodic"

$1 \Rightarrow n_x$ same

$2 \rightarrow$

centered in space at $n=1$

$\psi_2^n - \psi_{n_x-1}$