Time Differencing: Physically Insignificant Fast Waves

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A question of efficiency

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- The speed of sound is 300 m s^{-1}.

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- A fully explicit time differencing approximation to the full compressible governing equations will require the time step to be no larger than roughly 1 second.

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  - Semi-implicit methods
  - Splitting methods
  - The approximation of (high frequency) sound waves is wildly inaccurate

Outline

The Projection Method
- Boussinesq context
- Solution procedure

The Semi-Implicit Method
- Large time steps and poor accuracy
- The oscillation equation
- The shallow-water equations
- Stratified flow

Splitting Methods
- Complete operator splitting
- Partial operator splitting

Summary

Boussinesq equations

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla P = \mathbf{F} \equiv -\mathbf{v} \cdot \nabla \mathbf{v} + b\mathbf{k},$$

$$\frac{db}{dt} + N^2 w = 0,$$

$$\nabla \cdot \mathbf{v} = 0.$$

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Here

- P is the Boussinesq pressure potential
- b is buoyancy
- N is the Brunt-Väisälä (buoyancy) frequency

Integrate momentum equation over Δt

$$\int_{t^n}^{t^{n+1}} \frac{\partial \mathbf{v}}{\partial t}\, dt = -\int_{t^n}^{t^{n+1}} \nabla P \, dt + \int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{v}, b)\, dt$$

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$$\mathbf{v}^{n+1} - \mathbf{v}^n = -\Delta t\, \nabla \tilde{P}^{n+1} + \int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{v}, b)\, dt.$$

Update the velocities in two steps

1) Compute $\tilde{\mathbf{v}}$, defined such that

$$\tilde{\mathbf{v}} = \mathbf{v}^n + \int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{v}, b)\, dt.$$

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2) After evaluating $\tilde{P}$, complete the update

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}} - \Delta t\, \nabla \tilde{P}^{n+1}.$$

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From $\nabla \cdot \mathbf{v}^{n+1} = 0$ and

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Specification of boundary conditions for this Poisson equation can be an nontrivial detail.

Interpretation as a projection

The operation

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}} - \Delta t \, \nabla \tilde{P}^{n+1},$$

projects the partially updated velocities $\tilde{\mathbf{v}}$ onto the subspace of non-divergent velocity fields.

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Similar methods can also be used to integrate related sound-proof systems

- The anelastic equations
- The pseudo-incompressible equations

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Summary

Semi-discrete approximations to the advection equation

Suppose the time derivative in

$$\frac{\partial\psi}{\partial t} + c\frac{\partial\psi}{\partial x} = 0$$

is approximated by leapfrog time differencing as

$$\frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} + c\left(\frac{\partial\phi}{\partial x}\right)^n = 0.$$

Leapfrog phase speed

Wave solutions of the form

$$\phi^n(x) = e^{i(kx - \omega n \Delta t)}$$

have phase speed

$$c_{\mathrm{lf}} = \frac{\omega}{k} = \frac{\arcsin(ck\Delta t)}{k\Delta t}.$$

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Stability requires

- ω real
- $|ck\Delta t| < 1$

Trapezoidal phase speed

Using trapezoidal time differencing (over $2\Delta t$)

$$\frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} + \frac{c}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^{n+1} + \left(\frac{\partial \phi}{\partial x}\right)^{n-1}\right] = 0$$

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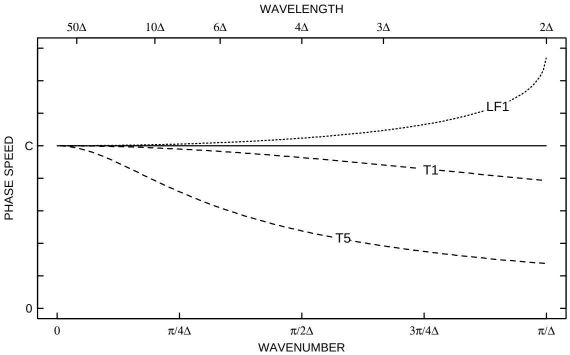
$$c_{\mathrm{t}} = \frac{\omega}{k} = \frac{\arctan(ck\Delta t)}{k\Delta t}.$$

- Unconditionally stable
- Phase speed reduced relative to that of the leapfrog solution by the factor $\cos(\omega\Delta t)$

Phase speed as a function of wavelength



Phase speed of leapfrog (dotted) and $2\Delta t$-trapezoidal (dashed) approximations to the advection equation when $c\Delta t/\Delta x = 1/\pi$ (LF1 and T1), and for the trapezoidal solution when $c\Delta t/\Delta x = 5/\pi$ (T5).

A prototype ODE

$$\frac{d\psi}{dt} + i\omega_H\psi + i\omega_L\psi = 0$$

Suppose the high frequency forcing exceeds that of the low frequency forcing $|\omega_L| < |\omega_H|$.

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$$\frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} + i\omega_H\left(\frac{\phi^{n+1} + \phi^{n-1}}{2}\right) + i\omega_L\phi^n = 0$$

Stability condition

Solutions of the form $\exp(-i\omega n\Delta t)$ exist provided

$$\sin(\omega\Delta t) = \omega_H \Delta t \cos(\omega\Delta t) + \omega_L \Delta t. \tag{1}$$

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- Always satisfied when $|\omega_L| < |\omega_H|$.
- Always satisfied when $|\omega_L \Delta t| < 1$.
  - Same criterion is obtained if $\omega_H = 0$
  - The terms approximated implicitly have no impact on stability

Linearized one-dimensional shallow-water system

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0,$$
$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + H\frac{\partial u}{\partial x} = 0,$$

- U and $u(x,t)$ are the mean and perturbation fluid velocity
- H and $\eta(x,t)$ are the mean and perturbation fluid depth

Numerical operator notation

Finite difference over interval $n\Delta t$

$$\delta_{nt} f(t) = \frac{f(t + n\Delta t/2) - f(t - n\Delta t/2)}{n\Delta t}$$

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Averaging operator over interval $n\Delta t$

$$\langle f(t) \rangle^{nt} = \frac{f(t + n\Delta t/2) + f(t - n\Delta t/2)}{2}$$

Semi-discrete approximation

Semi-implicit approximation

$$\delta_{2t} u^n + U \frac{du^n}{dx} + g \left\langle \frac{d\eta^n}{dx} \right\rangle^{2t} = 0,$$

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- Advection is leapfrog.
- Pressure gradient and divergence is trapezoidal over $2\Delta t$.

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- Same form as (1) for the oscillation equation
- Always stable if flow is subcritical ($|U| < \sqrt{gH}$)

Nonlinear one-dimensional shallow-water system

$$\frac{\partial u}{\partial t} + (U + u)\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0,$$

$$\frac{\partial \eta}{\partial t} + (U + u)\frac{\partial \eta}{\partial x} + (H + \eta)\frac{\partial u}{\partial x} = 0.$$

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- Nonlinear part of the divergence is leapfrog.

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- Linearize about a basic state at rest with non-zero perturbation displacement $\overline{\eta}$.
- Stability requires $|\overline{\eta}| < H$, which seems easy to satisfy.
- Local phase speed $\sqrt{g(H + \overline{\eta}}$ cannot exceed the reference phase speed $\sqrt{gH}$ by more than $\sqrt{2}$.

Application to continuously stratified flow

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Choose the reference-state stratification as isothermal.

Compressible Boussinesq system

Boussinesq system with prognostic equation for pressure retained.

$$\frac{d\mathbf{v}}{dt} + \nabla P = b\mathbf{k},$$

$$\frac{db}{dt} + N^2 w = 0,$$

$$\frac{dP}{dt} + c_s^2 \nabla \cdot \mathbf{v} = 0.$$

(Simple system supporting both sound and gravity waves.)

Semi-implicit formulation

$$\mathbf{v}^{n+1} + \Delta t \nabla P^{n+1} = \hat{\mathbf{v}} \equiv \mathbf{v}^{n-1} - \Delta t \left[\nabla P^{n-1} - 2b^n \mathbf{k} + 2\mathbf{v}^n \cdot \nabla \mathbf{v}^n \right]$$

$$P^{n+1} + c_s^2 \Delta t \nabla \cdot \mathbf{v}^{n+1} = h \equiv P^{n-1} - c_s^2 \Delta t \left[\nabla \cdot \mathbf{v}^{n-1} + 2\mathbf{v}^n \cdot \nabla P^n \right].$$

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Substituting the divergence of the first into the second yields

$$\nabla^2 P^{n+1} - \frac{P^{n+1}}{(c_s \Delta t)^2} = \frac{\nabla \cdot \hat{\mathbf{v}}}{\Delta t} - \frac{h}{(c_s \Delta t)^2}$$

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Compare with corresponding result from the projection method

$$\nabla^2 \tilde{p}^{n+1} = \frac{\nabla \cdot \tilde{\mathbf{v}}}{\Delta t}.$$

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Two fractional steps

A partial differential equation of the general form

$$\frac{\partial \psi}{\partial t} + \mathcal{L}_1(\psi) + \mathcal{L}_2(\psi) = 0,$$

may be approximated using two fractional steps as

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$$\phi^{n+1} = \mathcal{F}_2(\Delta t)\phi^s.$$

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- $\mathcal{F}_i(\Delta t)\phi^n$ approximates the action of the exact operator $\mathcal{L}_i$ mapping $\psi(t_n)$ to $\psi(t_{n+1})$.
- If $\mathcal{L}_i$ is a time-independent linear operator

$$\psi(t+\Delta t) = \exp(\Delta t \mathcal{L}_i)\psi(t) = \left(I + \Delta t \mathcal{L}_i + \frac{(\Delta t)^2}{2}\mathcal{L}_i^2 + \dots \right) \psi(t).$$

Unequal time steps

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A first-order splitting (unless the operators commute) is

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A first-order splitting (unless the operators commute) is

$$\phi^{n+1} = [\mathcal{F}_2(\Delta t/M)]^M \, \mathcal{F}_1(\Delta t)\phi^n.$$

A second-order (Strang) splitting is

$$\phi^{n+1} = [\mathcal{F}_2(2\Delta t/M)]^{(M/2)} \, \mathcal{F}_1(\Delta t) \, [\mathcal{F}_2(2\Delta t/M)]^{(M/2)} \, \phi^n.$$

A test case

Compressible 2D Boussinesq system

$$\frac{\partial \mathbf{r}}{\partial t} + \mathcal{L}_1(\mathbf{r}) + \mathcal{L}_2(\mathbf{r}) = \mathbf{F}(x, z, t),$$

where $\mathbf{r} = \begin{pmatrix} u & w & b & P \end{pmatrix}^T$ and a nondivergent forcing is applied to the momentum equations

$$\mathbf{F} = \begin{pmatrix} -\frac{\partial \Psi}{\partial z} & \frac{\partial \Psi}{\partial x} & 0 & 0 \end{pmatrix}^T.$$

Here Ψ is a streamfunction with compact support.

Advection step

Use linearly 3rd-order Runge-Kutta scheme and a large time step to advance the advection operator

$$\mathcal{L}_1 = \begin{pmatrix} \mathbf{v}\cdot\nabla & 0 & 0 & 0 \\ 0 & \mathbf{v}\cdot\nabla & 0 & 0 \\ 0 & 0 & \mathbf{v}\cdot\nabla & 0 \\ 0 & 0 & 0 & \mathbf{v}\cdot\nabla \end{pmatrix}$$

Pressure gradient, divergence and buoyancy

Use a small time step and forward-backward differencing, except for terms related to vertical propagation of sound waves, which are trapezoidal to improve stability.

Pressure gradient, divergence and buoyancy

Use a small time step and forward-backward differencing, except for terms related to vertical propagation of sound waves, which are trapezoidal to improve stability.

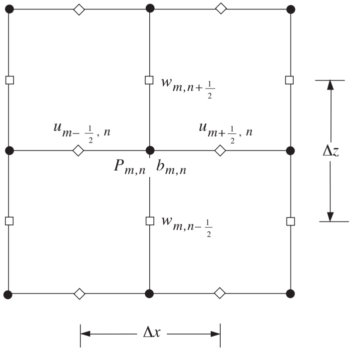
$$\frac{u^{m+1} - u^m}{\Delta\tau} + \frac{\partial P^m}{\partial x} = -\frac{\partial \Psi^{m+\frac{1}{2}}}{\partial z},$$

$$\frac{w^{m+1} - w^m}{\Delta\tau} + \frac{\partial}{\partial z}\left(\frac{P^{m+1} + P^m}{2}\right) - b^m = \frac{\partial \Psi^{m+\frac{1}{2}}}{\partial x},$$

$$\frac{b^{m+1} - b^m}{\Delta\tau} + N^2 w^{m+1} = 0,$$

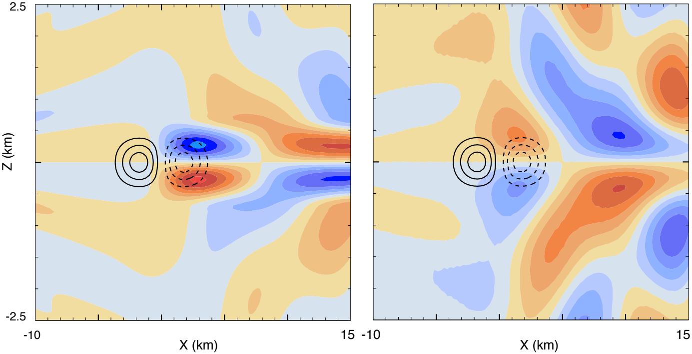
$$\frac{P^{m+1} - P^m}{\Delta\tau} + c_s^2 \frac{\partial u^{m+1}}{\partial x} + c_s^2 \frac{\partial}{\partial z}\left(\frac{w^{m+1} + w^m}{2}\right) = 0,$$

More details



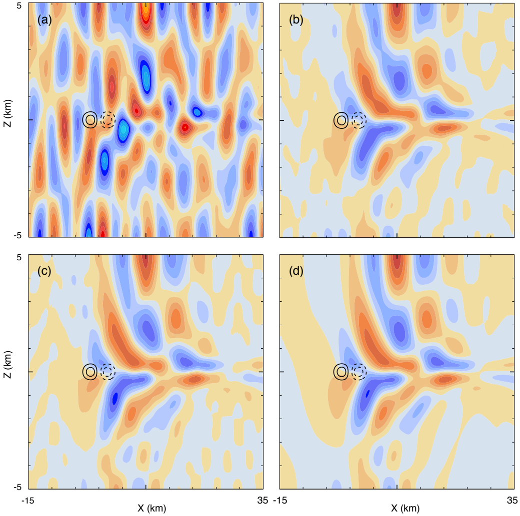
- Centered 2nd-order spatial differences on staggered mesh
- Approximation to $\exp(\Delta t \mathcal{L}_1)$ is stable for $|U|\Delta t/\Delta x < 1.73$.
- Approximation to $\exp(\Delta \tau \mathcal{L}_2)$ is stable and non-damping for $\max(c_s/\Delta x, N)\Delta\tau < 1$.

Spatially uniform U



Contours of (a) $U + u$ and Ψ; (b) P and Ψ. No zero contour is drawn. Minor tick marks indicate the location of the P points on the numerical grid. Only the central portion of the domain is shown.

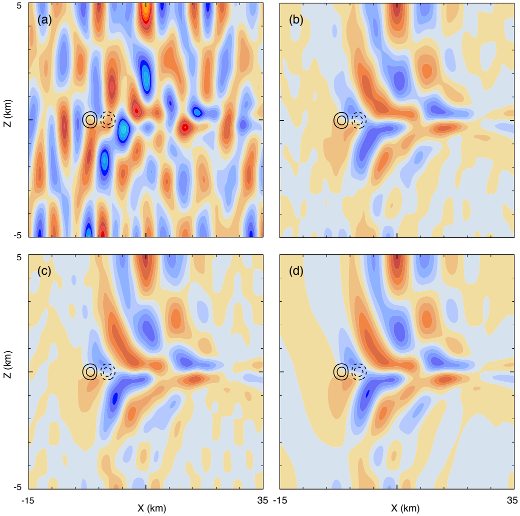
Vertically sheared U



Contours of P and Ψ. Tick marks every 20 grid intervals

(a) same Δt and $\Delta\tau$

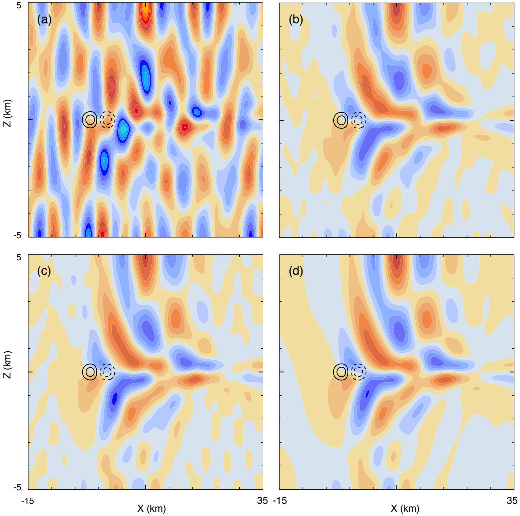
Vertically sheared U



Contours of P and Ψ. Tick marks every 20 grid intervals

(a) same Δt and $\Delta\tau$
(b) Both Δt and $\Delta\tau$ halved

Vertically sheared U



Contours of P and Ψ. Tick marks every 20 grid intervals

(a) same Δt and $\Delta\tau$
(b) Both Δt and $\Delta\tau$ halved
(c) Only Δt halved

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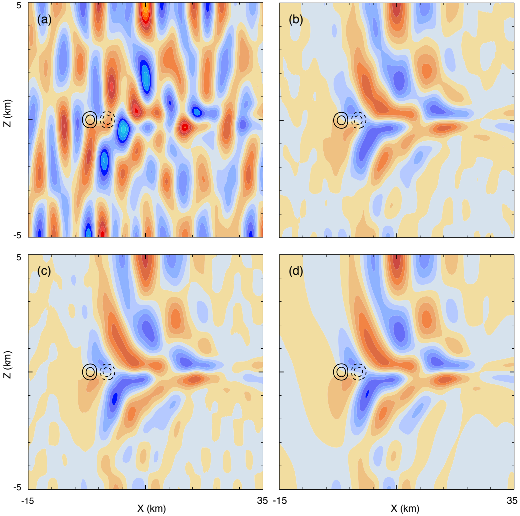

Contours of P and Ψ. Tick marks every 20 grid intervals

(a) same Δt and $\Delta\tau$
(b) Both Δt and $\Delta\tau$ halved
(c) Only Δt halved
(d) Partially split: same Δt and $\Delta\tau$

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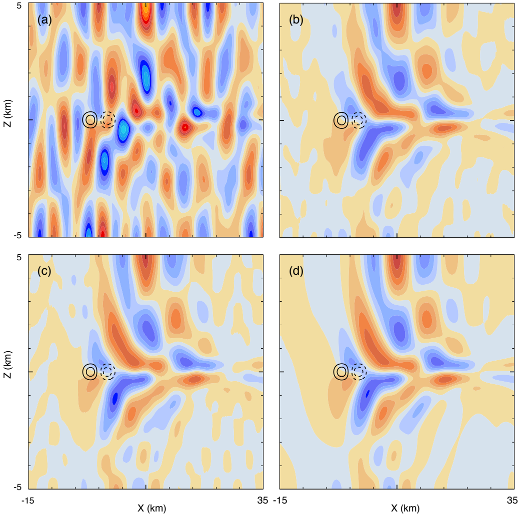

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(a) same Δt and $\Delta\tau$
(b) Both Δt and $\Delta\tau$ halved
(c) Only Δt halved
(d) Partially split: same Δt and $\Delta\tau$

In all cases $U + u$ remains correct at this time.

Partial splitting

Include a piece of the advective forcing on every small time step.

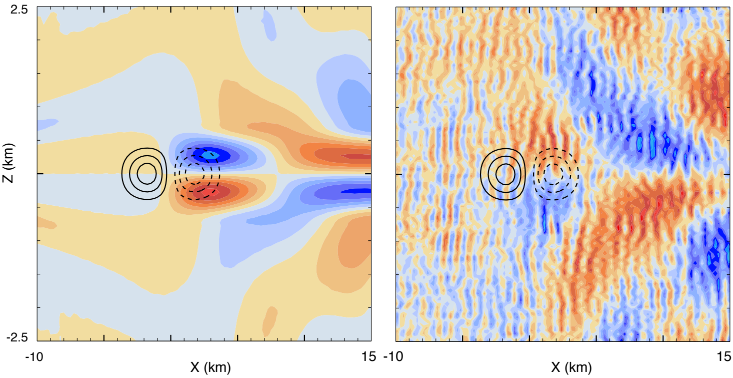
$$\frac{u^{m+1}-u^m}{\Delta\tau}+\frac{\partial P^m}{\partial x}=-U\frac{\partial u^n}{\partial x}-w^n\frac{\partial U}{\partial z},$$

$$\frac{w^{m+1}-w^m}{\Delta\tau}+\frac{\partial}{\partial z}\left(\frac{P^{m+1}+P^m}{2}\right)-b^m=-U\frac{\partial w^n}{\partial x},$$

$$\frac{b^{m+1}-b^m}{\Delta\tau}+N^2w^{m+1}=-U\frac{\partial b^n}{\partial x},$$

$$\frac{P^{m+1}-P^m}{\Delta\tau}+c_s^2\frac{\partial u^{m+1}}{\partial x}+c_s^2\frac{\partial}{\partial z}\left(\frac{w^{m+1}+w^m}{2}\right)=-U\frac{\partial P^n}{\partial x}.$$

Spatially uniform U – revisited



Contours of (a) $U + u$ and Ψ; (b) P (contoured using twice the interval as in the completely split case) and Ψ.

Fixing the partially split method

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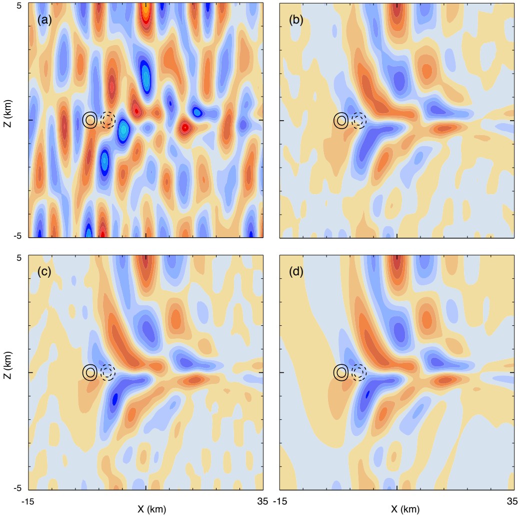
It can be stabilized using

- The Asselin time filter (used to prevent time-splitting in leapfrog integrations) (Tatsumi, 1983)
- Damping the velocity divergence every small time step (Skamarock and Klemp, 1992)

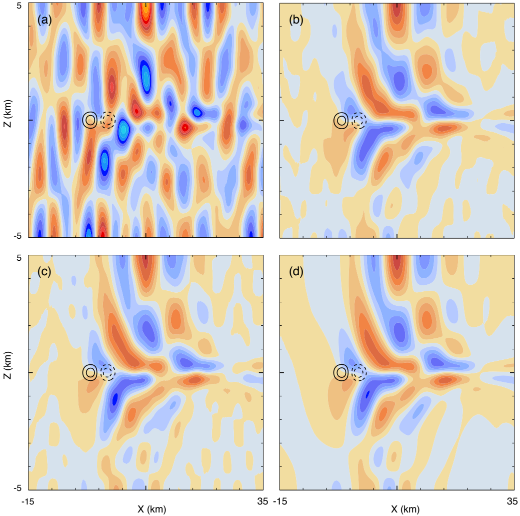
Comparison of complete and partial splitting – revisited



Contours of P and Ψ. Tick marks every 20 grid intervals. $\partial U/\partial z \neq 0$.

(a) Complete splitting, pressure noisy.

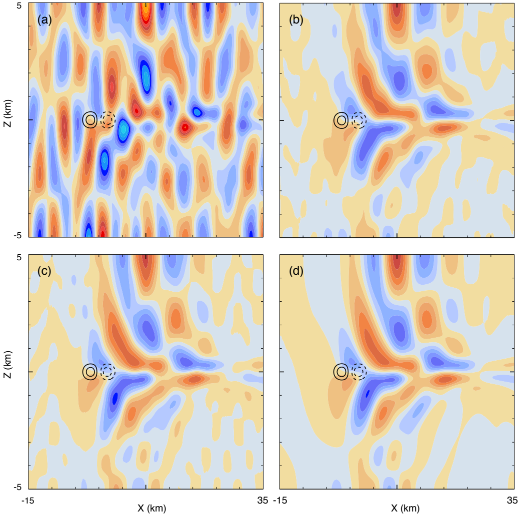
Comparison of complete and partial splitting – revisited



Contours of P and Ψ. Tick marks every 20 grid intervals. $\partial U/\partial z \neq 0$.

(a) Complete splitting, pressure noisy.
(b) Both Δt and $\Delta\tau$ halved

Comparison of complete and partial splitting – revisited



Contours of P and Ψ. Tick marks every 20 grid intervals. $\partial U/\partial z \neq 0$.

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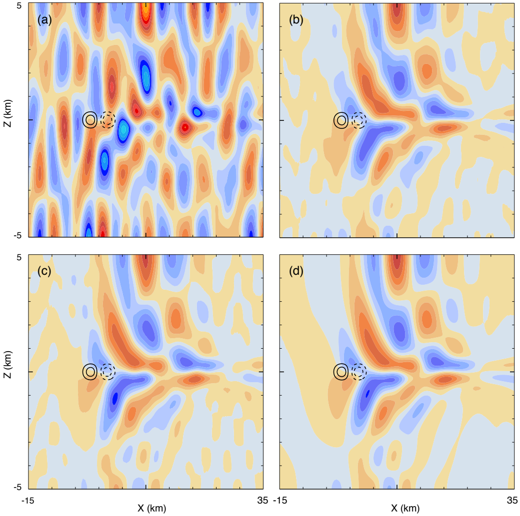

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(a) Complete splitting, pressure noisy.
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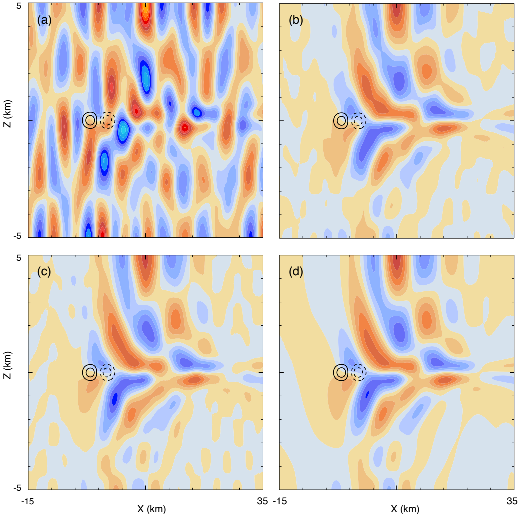

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Divergence damping does not significantly improve the completely split solutions.

Outline

The Projection Method
- Boussinesq context
- Solution procedure

The Semi-Implicit Method
- Large time steps and poor accuracy
- The oscillation equation
- The shallow-water equations
- Stratified flow

Splitting Methods
- Complete operator splitting
- Partial operator splitting

Summary

Projection method

The pressure P and divergence δ in the compressible Boussinesq system satisfy

$$\frac{\partial P}{\partial t} + c_s^2 \delta = -\mathbf{v} \cdot \nabla P,$$

$$\frac{\partial \delta}{\partial t} + \nabla^2 P = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + \frac{\partial b}{\partial z}.$$

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- Discard prognostic equation for P.
- Close 2nd equation by settting $\delta = 0$, and solve Poisson equation for P.
- Wide range of choices for integrating remaining terms

Semi-implicit method

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- Solve Helmholtz equation for P.

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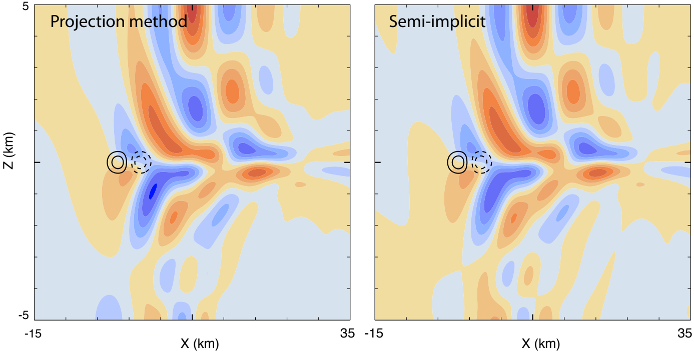
- Solve Helmholtz equation for P.
- Leapfrog differencing for integrating remaining terms

Test case with shear

Contours of P and Ψ as in previous.



Both the Projection and Semi-Implicit methods yield results similar to that obtained with partial splitting and divergence damping.

Complete splitting

During the small-time-step cycle, the divergence satisfies

$$\frac{\partial^2 \delta}{\partial t^2} - c_s^2 \nabla^2 \delta = \frac{\partial^2 b}{\partial t \partial z}.$$

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- Divergence generated on the large time step is added to the initial condition for δ at the start of each cycle, and builds throughout the integration.
- Advection by constant U is a Galilean translation that does not generate divergence.
- Divergence is generated when there is vertical shear.

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- Only small changes in δ are forced during a single large time step.
- The divergence forced over each small step is similar to that which would appear in an integration of the full compressible system provided the amplitudes of the sound waves are small compared to those of the slower modes.
- Divergence damping provides a feedback keeping sound-wave amplitudes small.

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Iterative solvers versus small steps

- The Poisson equations arising in the Boussinesq system could be solved by iteration.
- The inaccuracy of the completely split method suggests the small step cycle does not constitute an iteration process capable of arriving at the pressure that correctly projects the evolving velocity field onto the nondivergent (or anelastic equivalent) subspace.
- The partial splitting method succeeds because the forcing for the divergence stays close to that actually associated with the propagation of the slow modes.

References

Durran, D.R., 1999: *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics*, Springer. (Sections 7.1—7.4)

Skamarock, W.C. and J.B. Klemp, 1992: The stability of time-split numerical methods for the hydrostatic and nonhydrostatic elastic equations. *Mon. Wea. Rev.*, **120,** 2109-2171.