4.3 POTENTIAL VORTICITY (ERTEL ~~ISENTROPIC~~)

We know for inviscid flow

- Kelvins circulation theorem

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho}$$

- Vorticity equation

$$\frac{D}{Dt}(\zeta+f) = (\text{stretching}) + (\text{twisting}) + (\text{solenoidal})$$

Can we find a quantity related to fluid rotation and angular momentum which is conserved following the flow? Yes potential vorticity.

Potential vorticity conservation for fluids is the analog to angular momentum conservation for solid bodies.

The Ertel P.V. is derived from two basic relations.

- conservation of mass
- conservation of circulation on an isentropic surface – for adiabatic frictionless (isentropic) flow.

Conservation of circulation on a constant θ surface in isentropic flow

$$\text{const} = \Theta = T\left(\frac{p}{p_s}\right)^{-R/c_p} = \left(\frac{p}{\rho R}\right)\left(\frac{p}{p_s}\right)^{-R/c_p}$$

thus $$\rho = \left[\frac{p_s^{R/c_p}}{R\Theta}\right] p^{1-R/c_p} = \frac{1}{C} p^{c_v/c_p}$$

C constant $= C$ for isentropic flow

since $$\frac{dp}{\rho} = C\,\frac{dp}{p^{c_v/c_p}} = C\, d\left(p^{1-c_v/c_p}\right)$$

$$\oint \frac{dp}{\rho} = C \oint d\left(p^{1-c_v/c_p}\right) = 0$$

Thus $\frac{D}{Dt} C_a = 0$ let $\vec{\omega}_a$ be the absolute 3D vorticity

by Stokes theorem

$$\frac{D}{Dt}\left[\iint_A \vec{\omega}_a \cdot \vec{n}\, dA\right] = 0$$

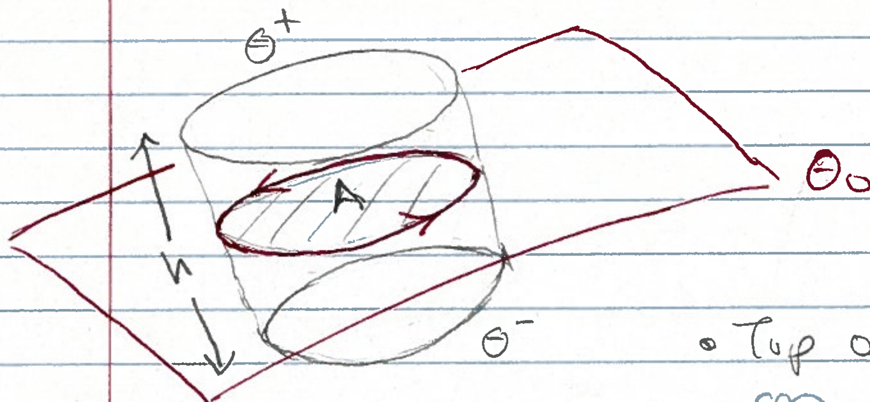
where $\vec{n}$ is the unit normal to the isentropic surface, which may be expressed as

$$\vec{n} = \frac{\nabla\theta}{|\nabla\theta|}$$

so
$$\boxed{\frac{D}{Dt}\left[\iint_A \vec{\omega}_a \cdot \frac{\nabla\theta}{|\nabla\theta|}\, dA\right] = 0}$$

Conservation of mass following a fluid parcel

Pick a fluid parcel bounded by constant θ surfaces that straddle the loop around which we compute the circulation



Circulation computed on θ_0 isentropic surface

- Top of volume embedded on θ^+ surface
- Bottom on θ^- surface
- Mass of fluid element is $\rho A h$

Note $\theta^+ - \theta^- = h\vec{n}\cdot\nabla\theta = h\dfrac{\nabla\theta}{|\nabla\theta|}\cdot\nabla\theta = h|\nabla\theta|$

solving for h: $h = \dfrac{\theta^+ - \theta^-}{|\nabla\theta|}$

Conservation of mass is $\dfrac{D}{Dt}(\rho A h) = 0$

or

$$\frac{D}{Dt}\left(\rho A \frac{(\theta^+ - \theta^-)}{|\nabla\theta|}\right) = 0 \Rightarrow \boxed{\frac{D}{Dt}\left(\frac{\rho A}{|\nabla\theta|}\right) = 0}$$

because θ^+ and θ^- are conserved following the flow $\dfrac{D\theta^+}{Dt} = \dfrac{D\theta^-}{Dt} = 0$

Now $\frac{D}{Dt}(C_a) = 0$ and $\frac{D}{Dt}(\text{Mass}) = 0$

for this fluid element, which implies $\frac{D}{Dt}\left(\frac{C_a}{\text{Mass}}\right) = 0$

or

$$\frac{D}{Dt}\left[\frac{\iint_A \left(\frac{\vec{\omega}_a \cdot \nabla\theta}{|\nabla\theta|}\right) dA}{\rho A \left(\frac{1}{|\nabla\theta|}\right)}\right] = 0$$

in the limit $A \to 0$

$$\frac{D}{Dt}\left[\frac{\vec{\omega}_a \cdot \nabla\theta \, \cancel{\frac{A}{|\nabla\theta|}}}{\rho \cancel{\frac{A}{|\nabla\theta|}}}\right] = 0$$

or $\frac{D}{Dt}\left[\frac{1}{\rho}\vec{\omega}_a \cdot \nabla\theta\right] = 0$

ERTEL P.V.
- defined at each point
- conserved following the flow for isentropic flow

More generally

$$\frac{D}{Dt}\left[\frac{1}{\rho}\vec{\omega}_a \cdot \nabla F\right] = 0$$

where F is any function

- of ρ & p only
- $\frac{DF}{Dt} = 0$ (F is conserved following the flow)

In 4.3. Holton derives an isentropic Ertel P.V. For large scale flow - in which only the vertical component contributes to $\vec{\omega}_a \cdot \nabla\theta$

that is $\frac{1}{\rho}\vec{\omega}_a \cdot \nabla\theta \approx \frac{1}{\rho}(\zeta + f)\frac{\partial\theta}{\partial z}$

recall $\Delta p = -\rho g \Delta z$ to excellent approximation,

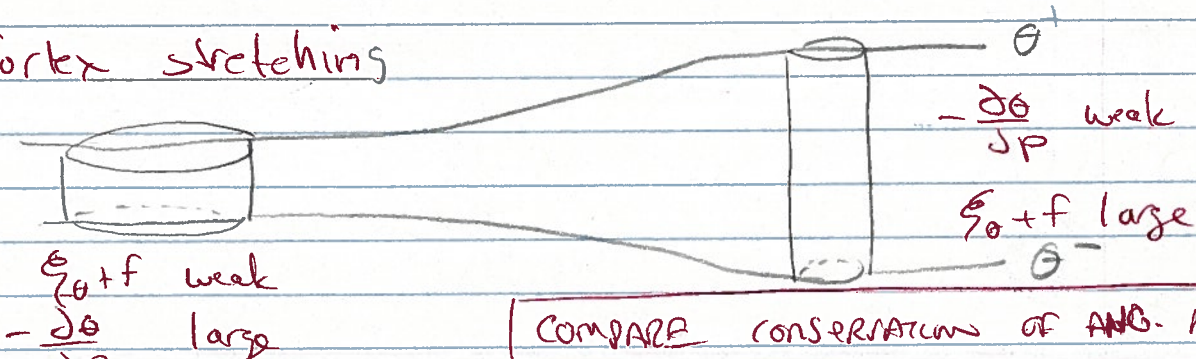
hence $\frac{1}{\rho}(\zeta + f)\frac{\partial\theta}{\partial z} = -g(\zeta + f)\frac{\partial\theta}{\partial p}$

assuming isentropes are approximately horizontal $\zeta \approx \zeta_\theta = \left.\frac{\partial v}{\partial x}\right|_\theta - \left.\frac{\partial u}{\partial y}\right|_\theta$

$$P = -(\zeta_\theta + f)\frac{\partial\theta}{\partial p}$$

For large-scale flow PV is often expressed in units of PVU $= 10^{-6}\,K\,kg^{-1}\,m^2\,s^{-1}$

Vortex stretching



COMPARE CONSERVATION OF ANG. MOMEN.