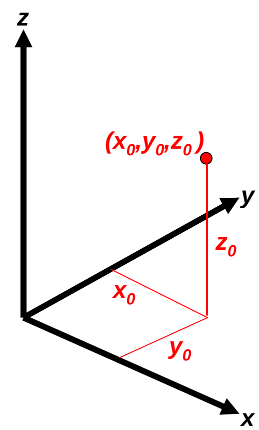
Coordinate Systems

- To describe the location in space of a point in a fluid, a coordinate system is used.
- A commonly used coordinate system is the rectangular, or x,y,z system (also known as Cartesian).



- Rectangular coordinates are often used to describe motions of the atmosphere or ocean, even though the earth is a sphere.
- In so doing, one assumes that the x-y plane is tangent to the surface of the spherical earth.
- General convention for use of rectangular coordinates:
  - x is a measure of distance from some origin and increases toward the east.
  - y is a measure of distance from some origin and increases toward the north.
  - z is zero at surface of earth and increases upward.

Fundamental Mathematical Concepts and Operations

- Fundamental state variables such as wind speed, temperature and pressure are functions of (i.e., depend upon) the independent variables (x, y, z, t).
- For example, atmospheric pressure can be expressed as a function of space and time:

$$P = P(x, y, z, t)$$

Derivatives

Assume Δx represents a small distance in the x direction.

The quotient $\frac{\Delta f}{\Delta x}$ represents the slope.

The derivative of a function $f(x)$ is defined as

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In the limit (as Δx goes to 0), this becomes the slope at a point and this is the derivative (df/dx), or the gradient or rate of change.

Partial Derivatives

With standard derivatives, our function varied in one dimension. However, some variables such as temperature vary not only in time, but also in space: $T(x, y, z, t)$

The partial derivative of T with respect to x will tell us how fast T changes as we move in the x direction and is defined as follows:

$$\frac{\partial T}{\partial x} = \lim_{\Delta x \to 0} \frac{T(x + \Delta x, y, z, t) - T(x, y, z, t)}{\Delta x}$$

Similarly,

$$\frac{\partial T}{\partial y} = \lim_{\Delta y \to 0} \frac{T(x, y + \Delta y, z, t) - T(x, y, z, t)}{\Delta y}$$

Chain Rule Of Differentiation

Assume:

$$f = f(u,v)$$
$$u = u(x,y)$$
$$v = v(x,y)$$

Then:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

More Identities

$$\frac{\partial(uv)}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

Order of partial differentiation does not matter.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial(\ln f)}{\partial x} = \frac{1}{f}\frac{\partial f}{\partial x}$$

Expansion of Total Derivative

If $f = f(x,y,z,t)$ then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

But $u \equiv \dfrac{dx}{dt}, \quad v \equiv \dfrac{dy}{dt}, \quad w \equiv \dfrac{dz}{dt}$

u = west-east component of fluid velocity
v = south-north component of fluid velocity
w = vertical component of fluid velocity

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\overset{u}{\left(\frac{dx}{dt}\right)} + \frac{\partial f}{\partial y}\overset{v}{\left(\frac{dy}{dt}\right)} + \frac{\partial f}{\partial z}\overset{w}{\left(\frac{dz}{dt}\right)}$$

Euler's relation (expansion of total derivative):

$$\underset{A}{\frac{df}{dt}} = \underset{B}{\frac{\partial f}{\partial t}} + \underset{C}{u\frac{\partial f}{\partial x}} + \underset{D}{v\frac{\partial f}{\partial y}} + \underset{E}{w\frac{\partial f}{\partial z}}$$

Term A: Total rate of change of *f* following the fluid motion

Term B: Local rate of change of *f* at a fixed location

Term C: Advection of *f* in *x* direction by the *x*-component flow

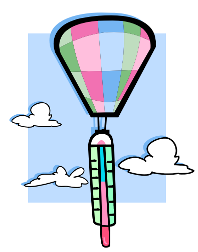
Term D: Advection of *f* in *y* direction by the *y*-component flow

Term E: Advection of *f* in *z* direction by the *z*-component flow

Total Derivative vs. Local Derivative

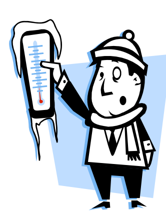
Total derivative is the temporal rate of change following the fluid motion. Example: A thermometer measuring changes as a balloon floats through the atmosphere.

$$\frac{dT}{dt}$$



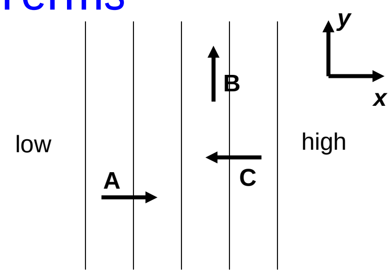
Local derivative is the temporal rate of change at a fixed point. Example: An observer measures changes in temperature at a weather station.

$$\frac{\partial T}{dt}$$



Advection Terms

Assume that thin lines are contours of a scalar quantity *f* and thick arrows indicate the fluid motion. We wish to evaluate the advection term $u\frac{\partial f}{\partial x}$



At point A: $u > 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} > 0$ ➡ **Transport from low to high: "negative advection of *f*"**

At point B: $u = 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} = 0$ ➡ **"neutral advection of *f*"**

At point C: $u < 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} < 0$ ➡ **Transport from high to low: "positive advection of *f*"**

Taylor Series

A function *f(x)* can be computed by Taylor expansion given the values of the function and its derivatives at a point x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$$

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

A truncated Taylor series can be used to approximate *f(x).*