Dynamics of a zonally symmetric vortex

Cartesian geometry
f -plane
zonal wind is geostrophic
meridional wind is ageostrophic
eddy fluxes, diabatic heating, and friction are specified

We will be able to determine
- MMC (diagnostic)
- future evolution of wind and temperature fields

The governing equations

Three equations
Three dependent variables
Boundary conditions
Initial conditions

$$\frac{\partial u}{\partial t} = -f\frac{\partial \psi}{\partial p} + G + F$$

$$\frac{\partial \alpha}{\partial t} = \sigma\frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial u}{\partial p} = \frac{1}{f}\frac{\partial \alpha}{\partial y}$$

temperature is replaced by specific volume

MMC are represented by a stream function

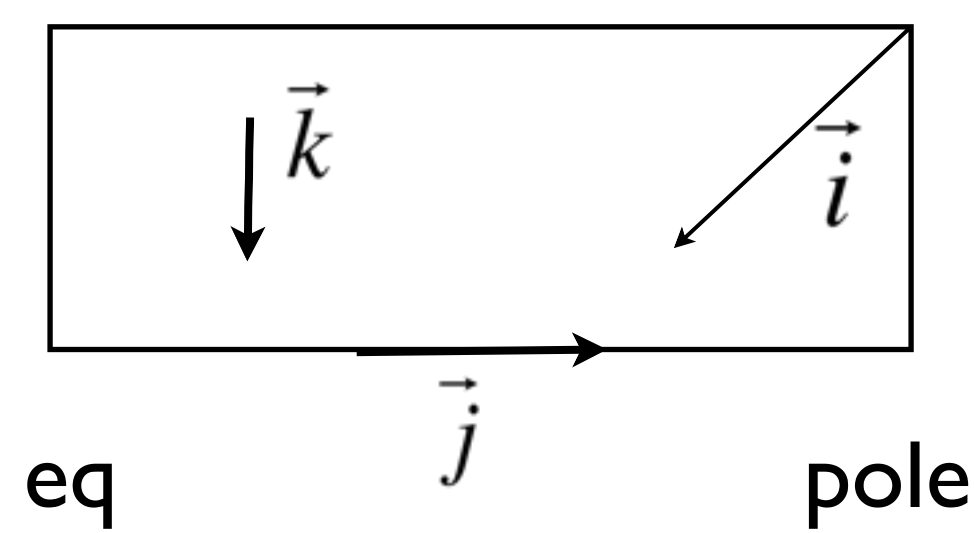
the pressure gradient force (PGF) in the meridional plane

$$\vec{\Sigma} = \left(\frac{\partial \hat{\Phi}}{\partial y} \vec{j},\ \frac{\partial \hat{\Phi}}{\partial p} \vec{k} \right) = \left(fu\, \vec{j},\ \hat{\alpha}\, \vec{k} \right)$$

geostrophic equation hypsometric equation

$$\vec{i} \cdot \left(\nabla \times \vec{\Sigma} \right) = 0$$

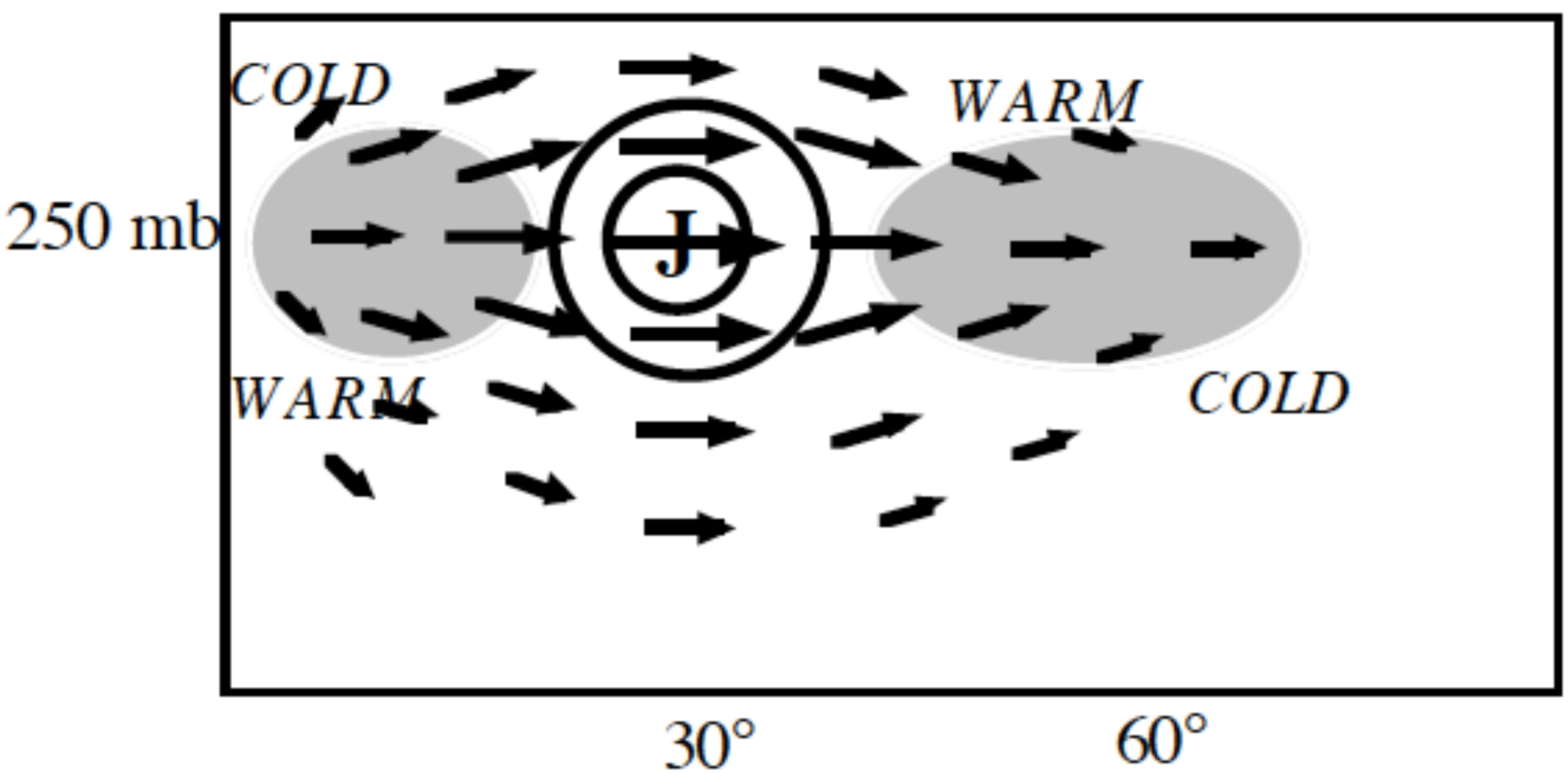
thermal wind equation:
the PGF is irrotational



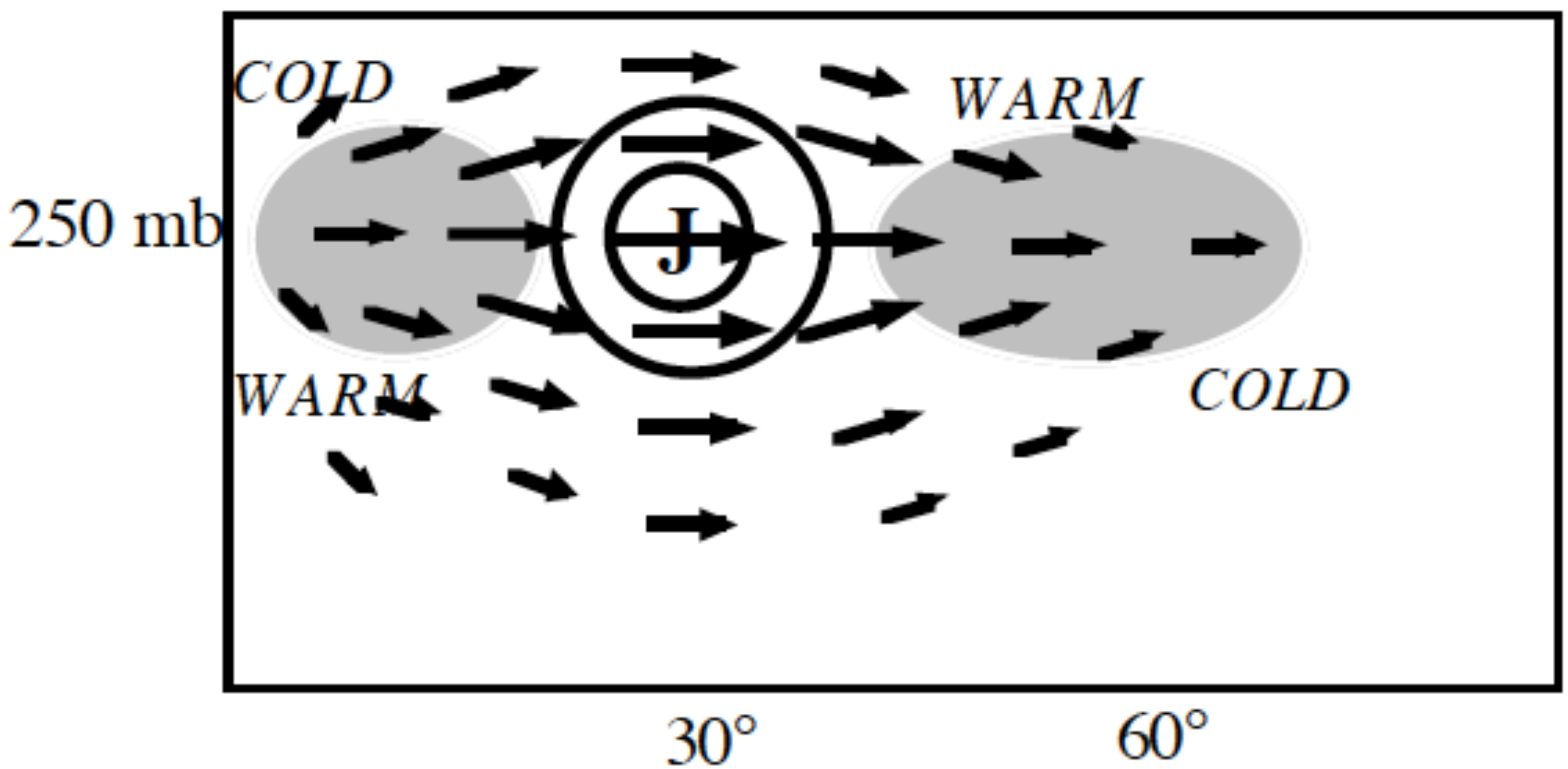
Concept of a stretched membrane

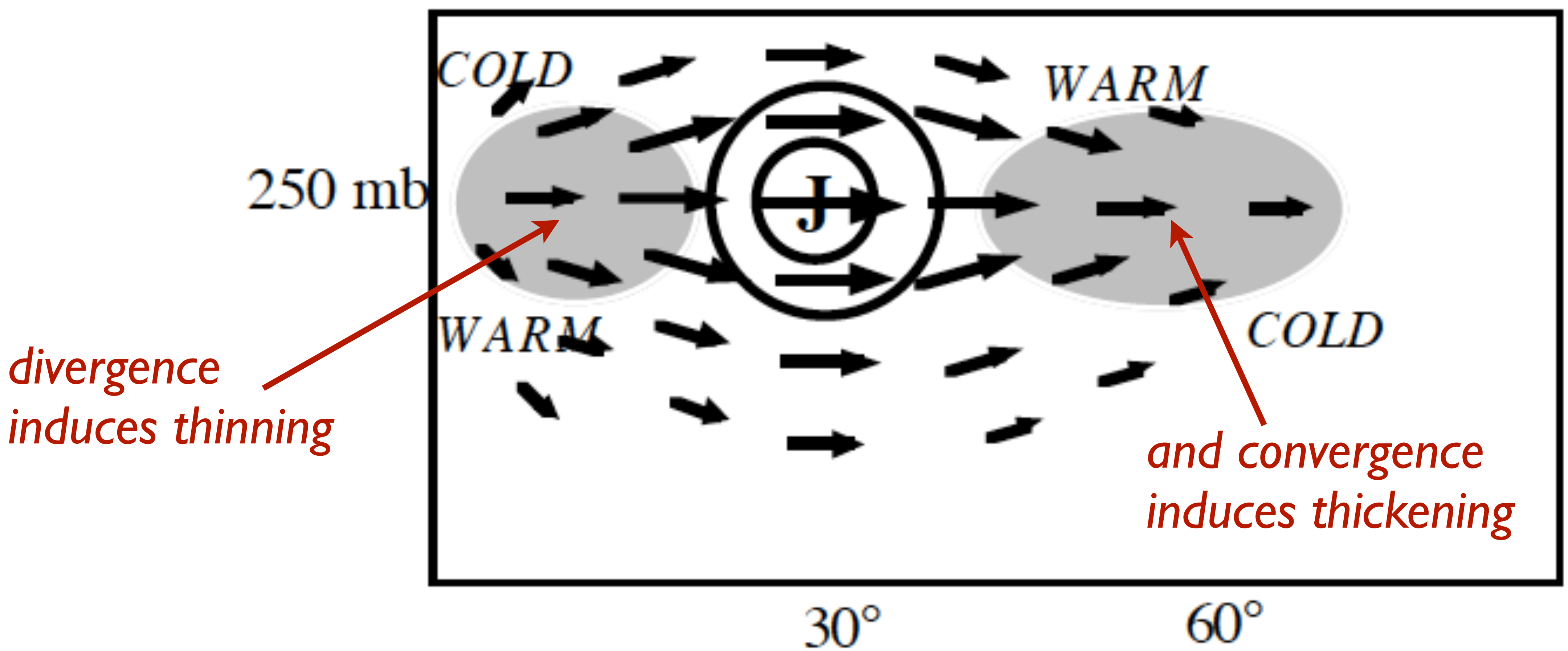
$$\vec{S} \equiv \left(\frac{u}{f}\vec{j},\ \frac{\widehat{\alpha}}{\sigma}\vec{k} \right)$$

the displacement vector



It is helpful to think of the displacement vector as relating to a stretched membrane. If the flow were at rest so that the displacement were everywhere equal to zero the thickness of the membrane would be uniform. Displacements act to thicken the membrane in some areas and thin it in others.

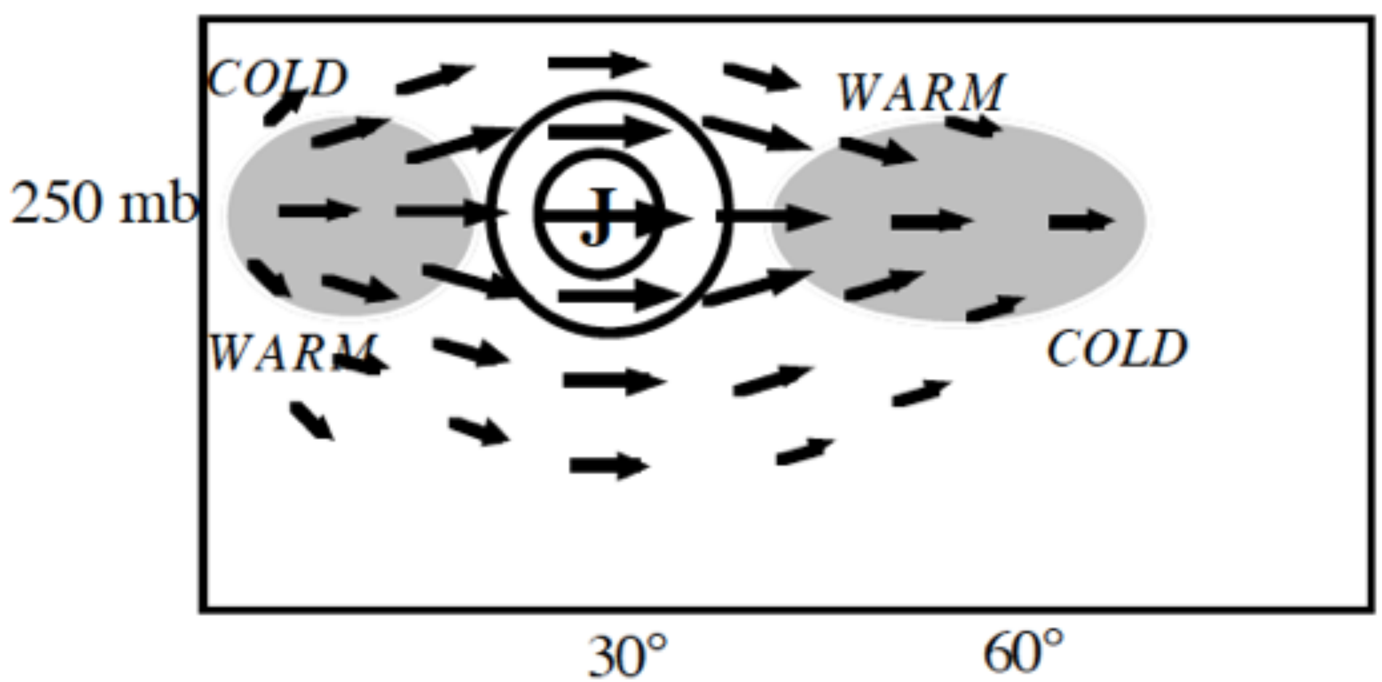



COLD
WARM
250 mb
J
WARM
COLD
divergence
induces thinning
and convergence
induces thickening
30°
60°

$$-\nabla \cdot S = -\frac{1}{f}\frac{\partial u}{\partial y} - \frac{\partial}{\partial p}\frac{\widehat{\alpha}}{\sigma}$$

$$-f\nabla \cdot S = -\frac{\partial u}{\partial y} - f\frac{\partial}{\partial p}\frac{\widehat{\alpha}}{\sigma}$$

quasi-geostrophic potential vorticity



so membrane mass is a measure of potential vorticity!

Modulus of elasticity

$$\vec{\Sigma} = fu\,\vec{j}, \hat{\alpha}\,\vec{k}$$

$$\vec{S} \equiv \left(\frac{u}{f}\vec{j}, \frac{\hat{\alpha}}{\sigma}\vec{k} \right)$$

$$\vec{\Sigma} = \left(f^2, \sigma\right)\vec{S}$$

"stiffness" of membrane with respect to horizontal and vertical displacements

$$\frac{1}{2}\vec{\Sigma}\cdot\vec{S} = \frac{u^2}{2} + \frac{\hat{\alpha}^2}{2\sigma} \qquad \frac{1}{2g}\int_0^{p_0} \overline{\vec{\Sigma}\cdot\vec{S}}\,dp = \overline{K} + \overline{A}$$

Additional definitions

the MMC vector

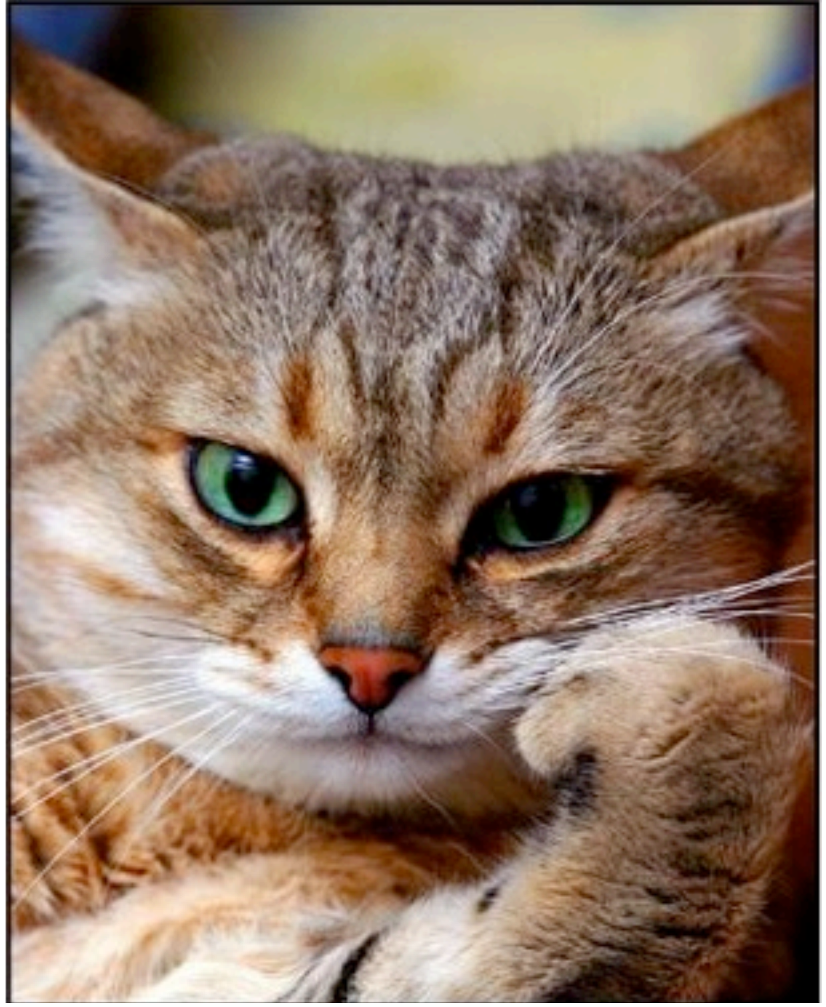
$$\vec{\Psi} \equiv \vec{i} \times \nabla \psi = \left(v\vec{j}, \omega\vec{k} \right)$$

the forcing vector

$$\vec{\Gamma} = \left(\frac{G+F}{f}\vec{j}, \frac{P+Q}{\sigma}\vec{k} \right)$$

The prognostic equation

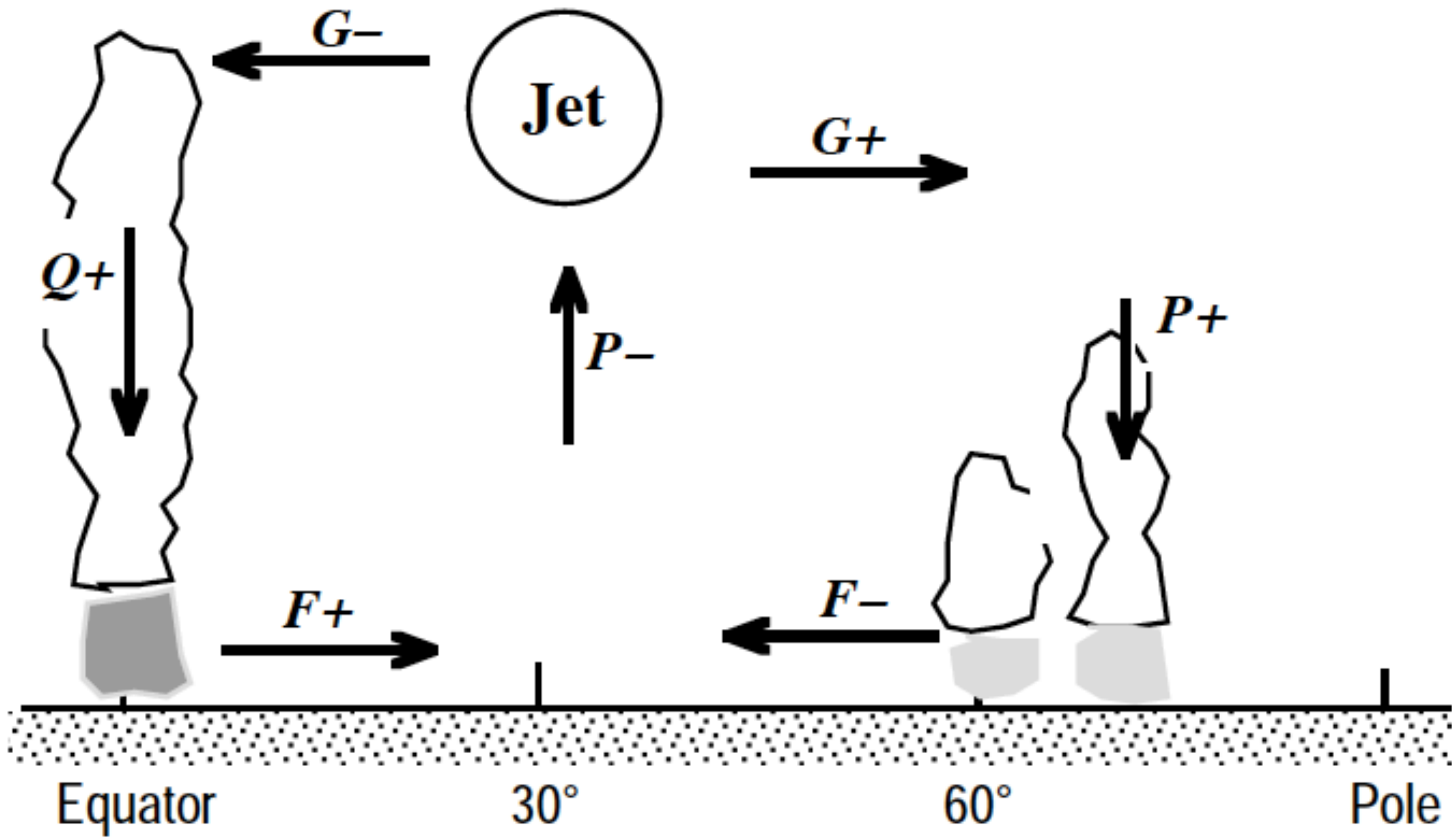
$$\frac{\partial \vec{S}}{\partial t} = \vec{\Gamma} + \vec{\Psi}$$



I’m skeptical. Can you give some evidence that all this mumbo jumbo makes sense?

It’s simple: consider a steady state....

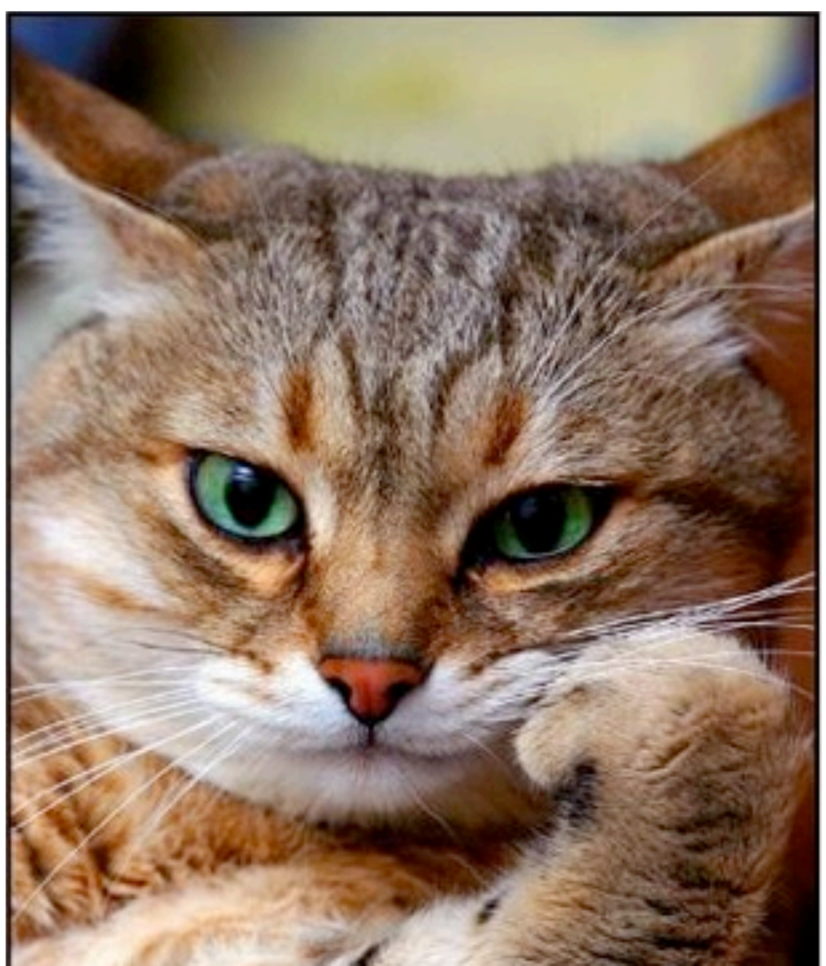
i.e., the climatological-mean flow



for steady state $\vec{\Psi} = -\vec{\Gamma}$

from which we can deduce the existence of the Hadley and Ferrell cells

Are you convinced?



But we already knew that.
What about the time varying case?

It’s simple. Just eliminate the time derivative terms from the governing equations and solve for the MMC.

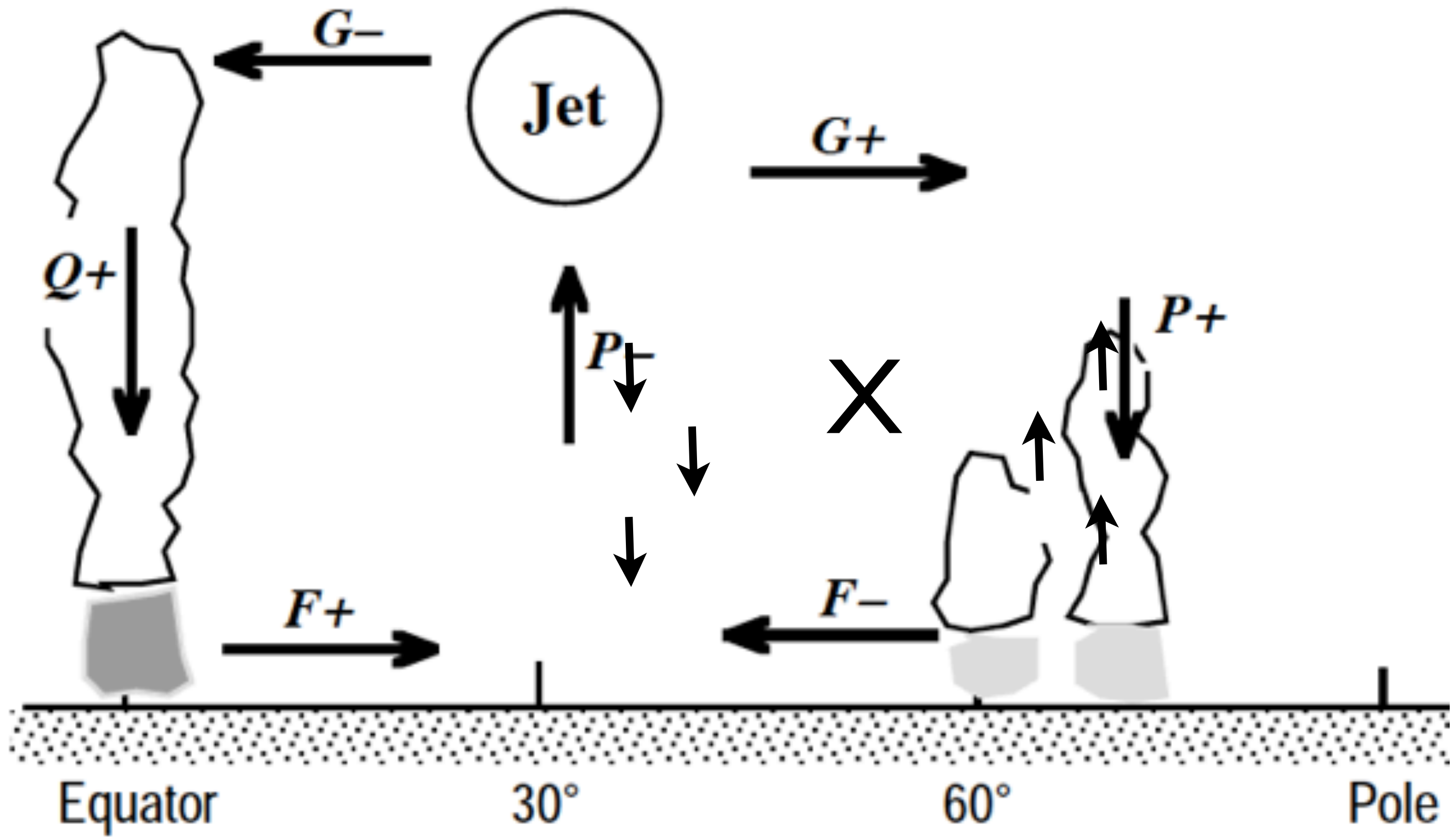
$$\frac{\partial u}{\partial t} = -f\frac{\partial \psi}{\partial p} + G + F$$

$$\frac{\partial \alpha}{\partial t} = \sigma\frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial u}{\partial p} = \frac{1}{f}\frac{\partial \alpha}{\partial y}$$

which yields

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$



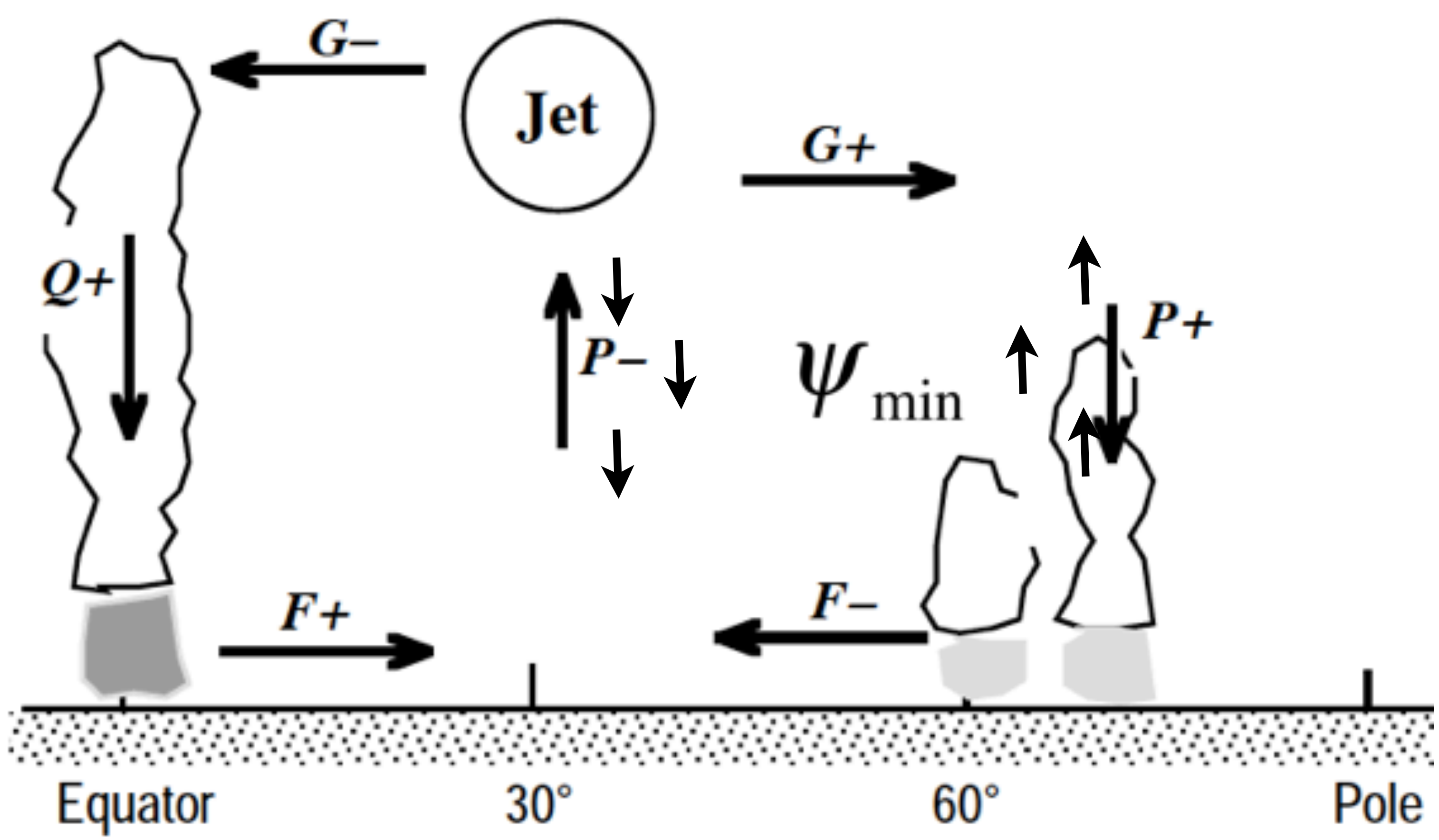
This is how it works for the Ferrell cell in steady state.

At Point X both terms on the RHS exhibit maxima; i.e.,

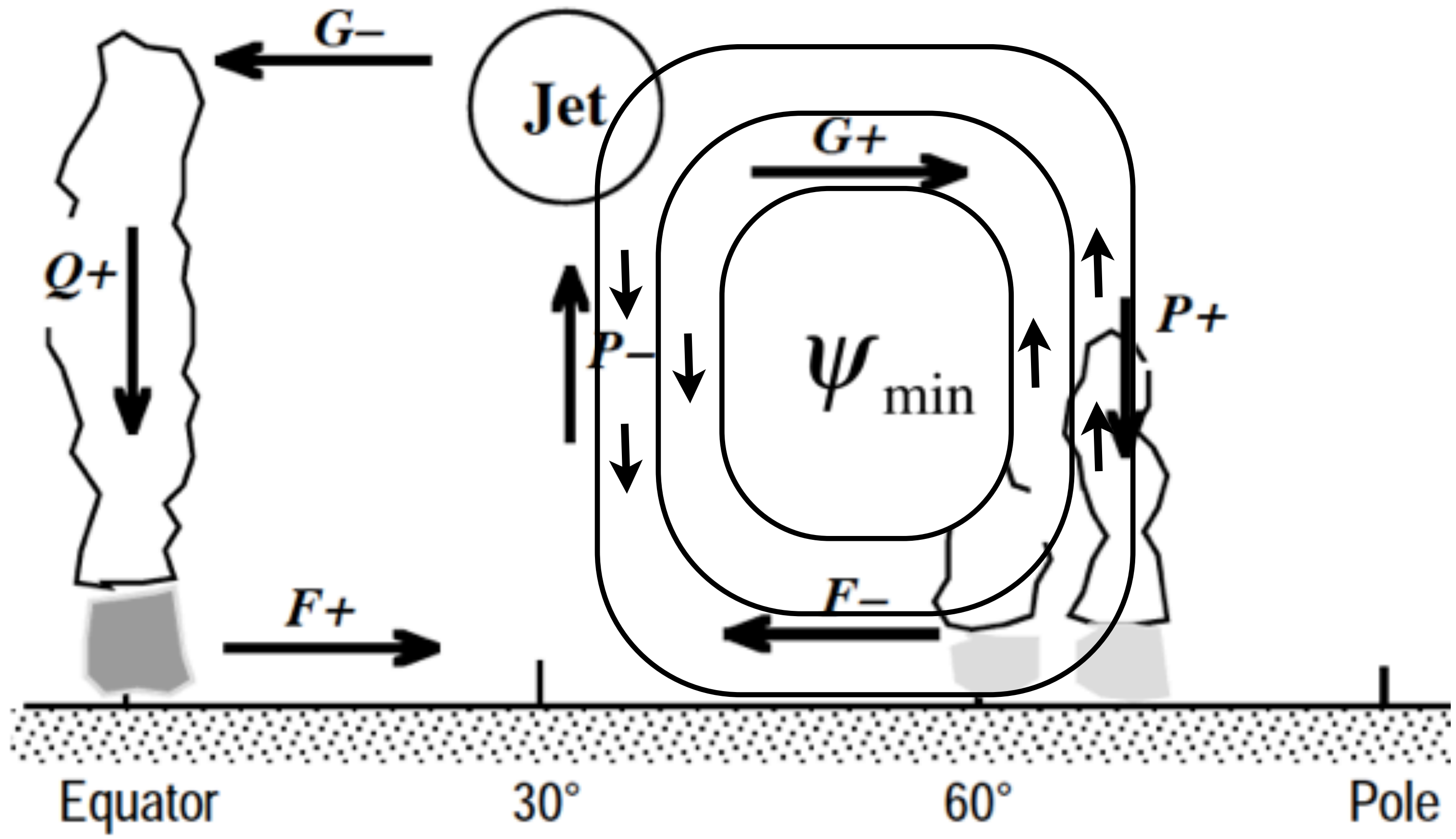
$G + F$ decreases with pressure

$P + Q$ increases with latitude

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

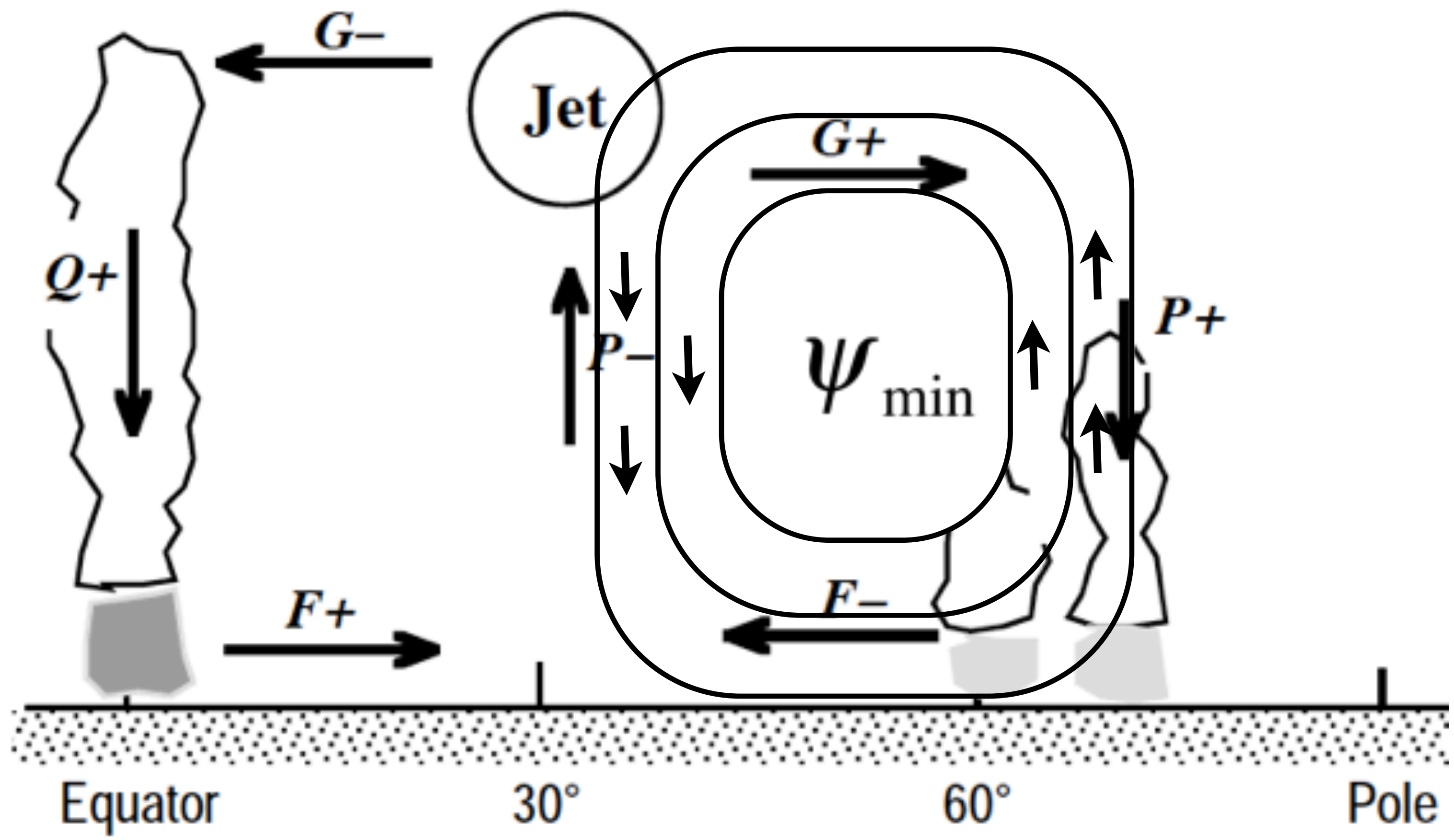


This is an elliptic equation, so where $A(\psi)$ exhibits a maximum, ψ exhibits a minimum.

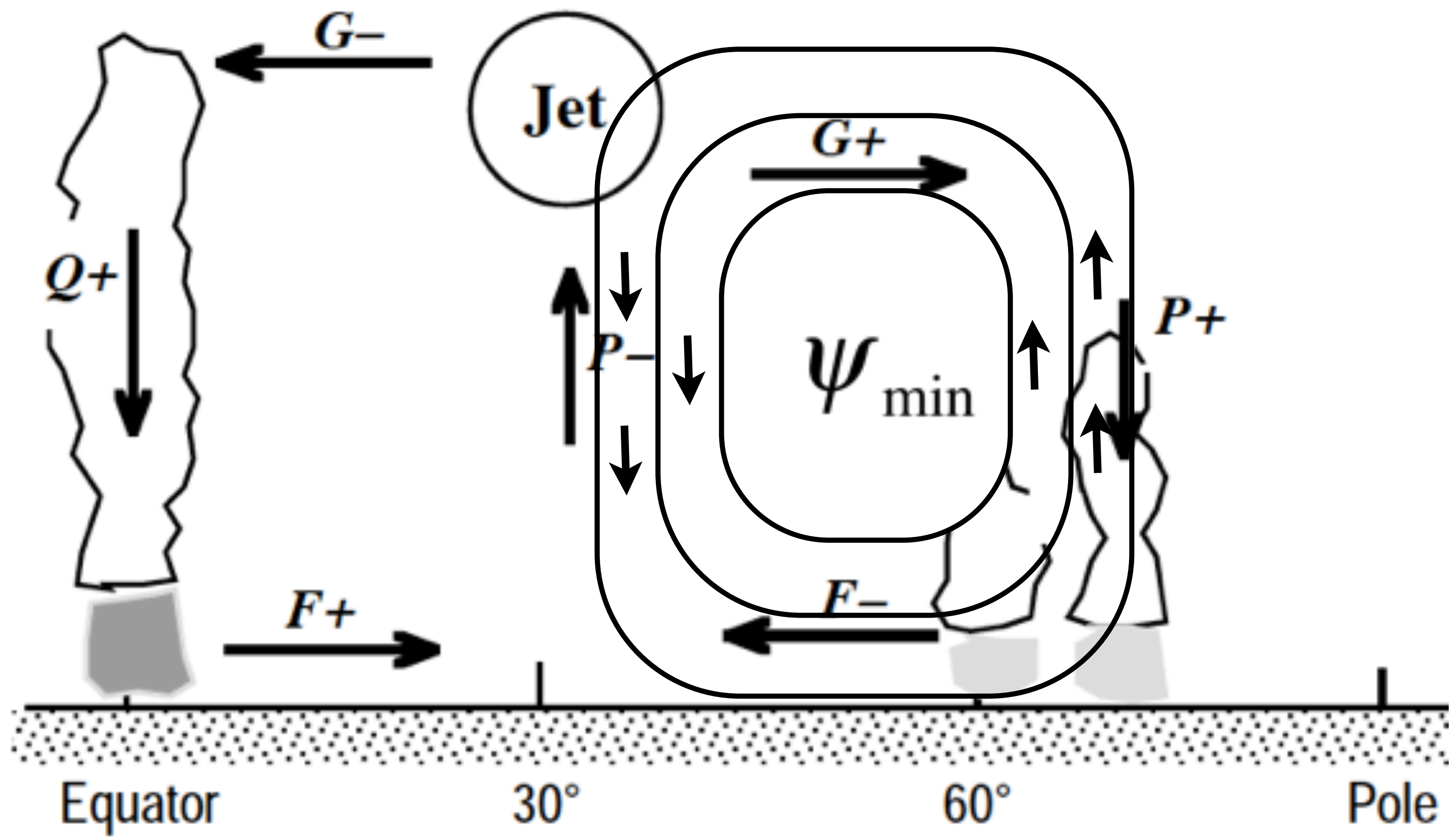


By convention $\vec{\Psi} \equiv \vec{i} \times \nabla \psi = \left(v\vec{j}, \omega\vec{k} \right)$

so the MMC circulate counterclockwise around ψ_{min}



Note that with the vectorial notation we can sometimes infer the sense of the MMC without considering consider the elliptic equation. In a case like this, where the curl of the forcing is obvious, the sense of the MMC is also obvious.



For the time-mean MMC, *d*/*dt* = 0 but the solution for ψ is generally valid .

Before we proceed, let's have a brief review:

There are four ways of inferring the MMC

1. direct measurement of $[v]$

2. vorticity balance $[\overline{v}] = -\dfrac{G+F}{f}$

3. total energy balance $[\overline{\omega}] = -\dfrac{P+Q}{\sigma}$

4. eliminating time derivatives in governing equations

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

Four ways of inferring the MMC

1. direct measurement of $[v]$ — it's a small residual

2. vorticity balance $[\overline{v}] = -\dfrac{G+F}{f}$ — doesn't address the time dependence

3. total energy balance $[\overline{\omega}] = -\dfrac{P+Q}{\sigma}$ — doesn't address the time dependence

4. eliminating time derivatives in governing equations

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

assumes geostrophy

There are four analogous ways of inferring ω in QG system

1. direct measurement of $\nabla \cdot \vec{V}$ it's a small residual

2. vorticity balance $\nabla \cdot \vec{V} = -\dfrac{\dfrac{\partial \varsigma}{\partial t} + \vec{V} \cdot \nabla \varsigma}{f + \varsigma}$ doesn't address the time dependence

3. total energy balance $\omega = -\dfrac{\dfrac{\partial T}{\partial t} + \vec{V} \cdot \nabla T}{\sigma}$ doesn't address the time dependence

4. eliminating time derivatives in governing equations

the omega equation assumes geostrophy

Now here's a vectorial interpretation of the elliptic equation for the MMC. The zonal wind and thickness fields need to stay in thermal wind balance so

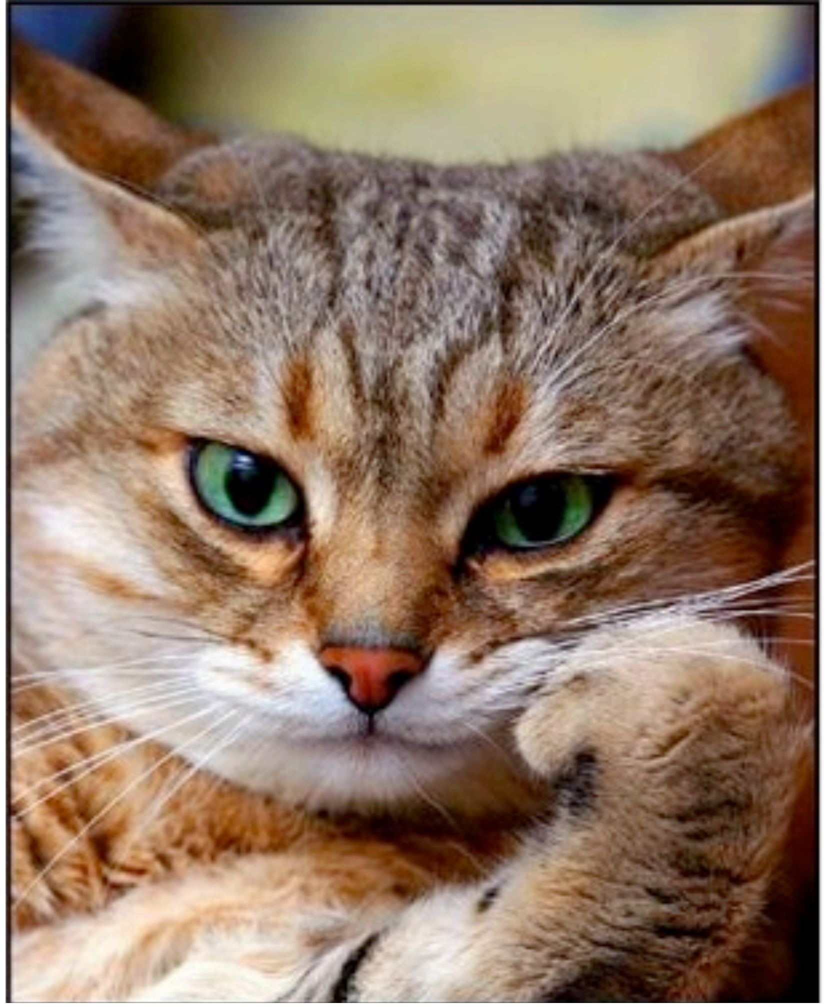
$$\frac{d}{dt}\left(\vec{i} \times \nabla\vec{\Sigma}\right) = 0$$

It follows that the curl of the tendency in $\vec{\Sigma}$ induced by the forcing vector must be balanced by the curl of the tendency induced by the MMC.

$\vec{\Sigma}$ can't be twisted because it's the gradient of a scalar

The solution for the MMC reduced to a statement about the curl of the pressure field! How can that be?

It’s simple. For 2D, zonally symmetric flow, the omega equation is the requirement that the zonal wind and temperature fields remain in thermal wind balance as they evolve. The MMC inferred from this equation ensure that they stay in thermal wind balance.



But what about the evolution of the zonally symmetric flow?

I'm glad you asked. It's simple. Once we've solved for the MMC we have everything we need to solve the prognostic equations

$$\frac{\partial u}{\partial t} = -f \frac{\partial \psi}{\partial p} + G + F$$

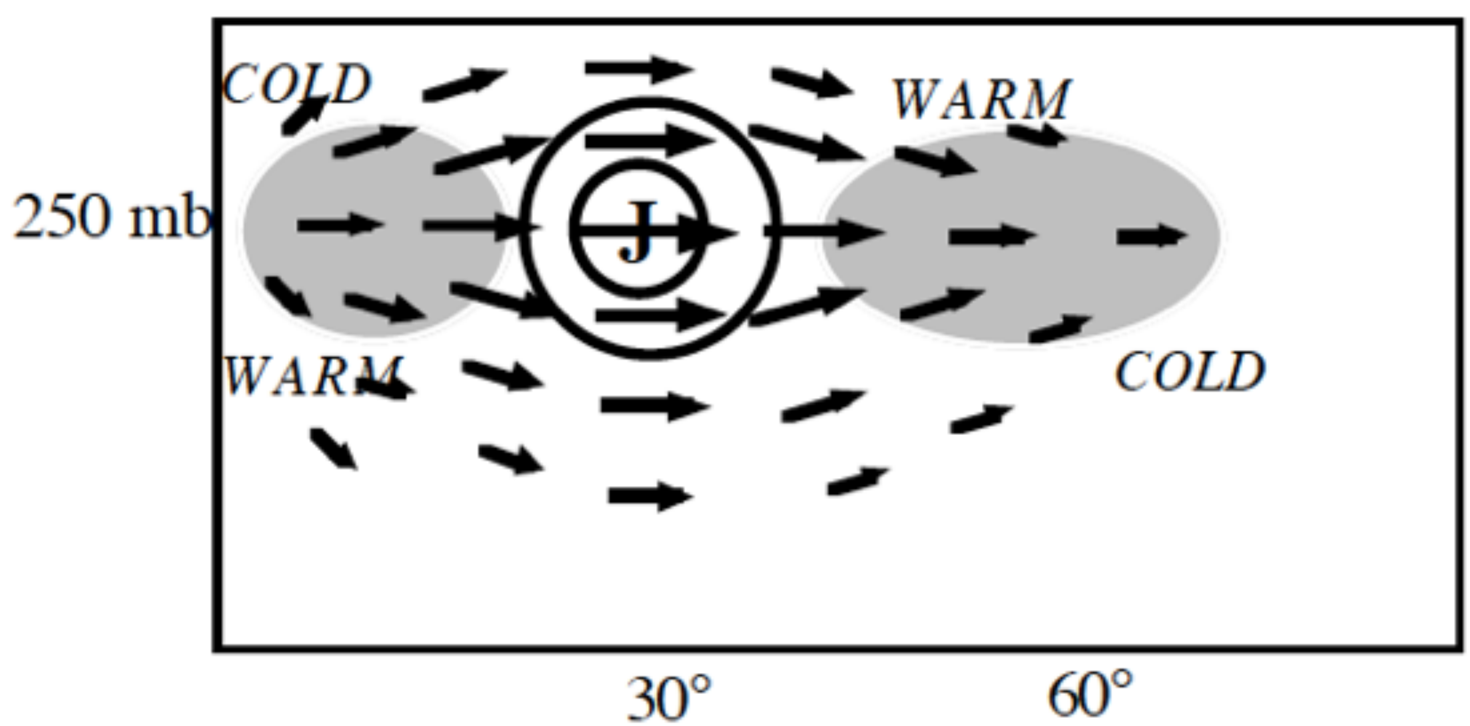
$$\frac{\partial \alpha}{\partial t} = \sigma \frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial \vec{S}}{\partial t} = \vec{\Gamma} + \vec{\Psi}$$

We can get insight into how the zonal flow evolves in response to the fluxes of heat and momentum by considering how the eddy fluxes of zonal momentum and temperature change the field of quasi-geostrophic potential vorticity q.

If we know how q is changing in response to the forcing, we can recover the fields of u, T, and Φ.

This is the so-called *invertibility principle*.



Recall that

$$q = -f\nabla \cdot \vec{S} = -\frac{\partial u}{\partial y} - f\frac{\partial}{\partial p}\frac{\hat{\alpha}}{\sigma}$$

and note that the MMC are nondivergent and therefore do not have any effect on potential vorticity or membrane mass.

It follows that

$$\frac{dq}{dt} = \frac{d}{dt}\left(-f\nabla \cdot \vec{S}\right) = -\nabla \cdot \vec{\Gamma} = -\frac{\partial}{\partial y}\frac{(G+F)}{f} - \frac{\partial}{\partial p}\frac{(P+Q)}{\sigma}$$

Interpretation of the forcing terms

$$\frac{dq}{dt} = -\frac{\partial}{\partial y}\frac{(G+F)}{f} - \frac{\partial}{\partial p}\frac{(P+Q)}{\sigma}$$

vorticity forcing

static stability forcing

meridional pinching

vertical pinching

Eddy fluxes of momentum moving vorticity around in the meridional plane?   How can that be?

It's simple. Held (*JAS 1975*) proved that G can also be interpreted as the poleward eddy flux of vorticity; i.e.,

$$G \equiv -\frac{\partial}{\partial y}[u * v *] = [\varsigma * v *]$$

You can prove it for yourself. Start with the expression for *G* in either of its forms and transform it into the other form. Following Held (1975) you will need to assume that the flow in the eddies is nondivergent; i.e., that

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0$$

$G = [\varsigma * v *]$ That's awesome!

Is there an analogous expression for the poleward flux of potential vorticity $[q * v *]$?

As a matter of fact, there is.
Allow me to explain.

$$\left[q^* v^*\right] = \left[\varsigma^* - f\frac{\partial}{\partial p}\frac{\alpha^*}{\sigma}\right]v^* = G - f\frac{\partial}{\partial p}\frac{\left[v^*\alpha^*\right]}{\sigma}$$

$$\frac{\partial}{\partial p}\left[v^*\alpha^*\right] = \left[v^*\frac{\partial \alpha^*}{\partial p}\right] + \left[\alpha^*\frac{\partial v^*}{\partial p}\right]$$

If we approximate v^* by $v_g{}^*$, and use the thermal wind equation, the second term on the right hand-side vanishes.

$$-\left[v^*\frac{\partial \alpha^*}{\partial p}\right]$$

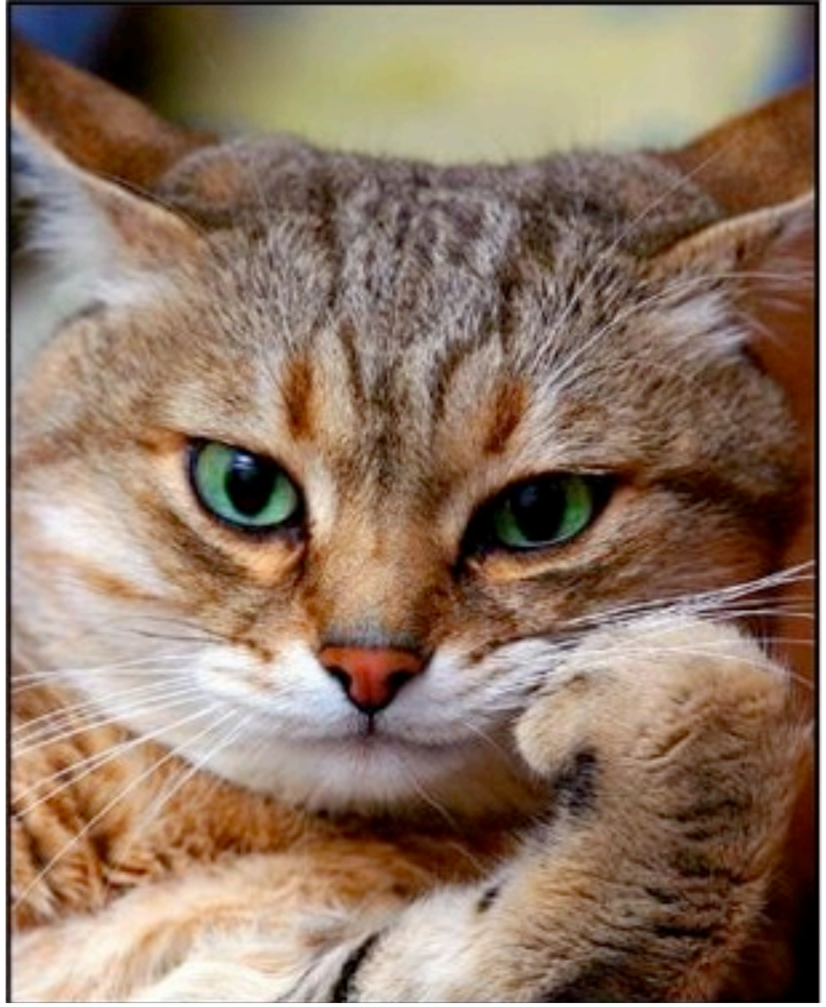
can be interpreted as
the poleward eddy flux of static stability

So to summarize,

$$[q^* v^*] = [\varsigma^* v^*] - f \frac{\partial}{\partial p} \frac{[v^* \alpha^*]}{\sigma}$$

vorticity flux

static stability flux

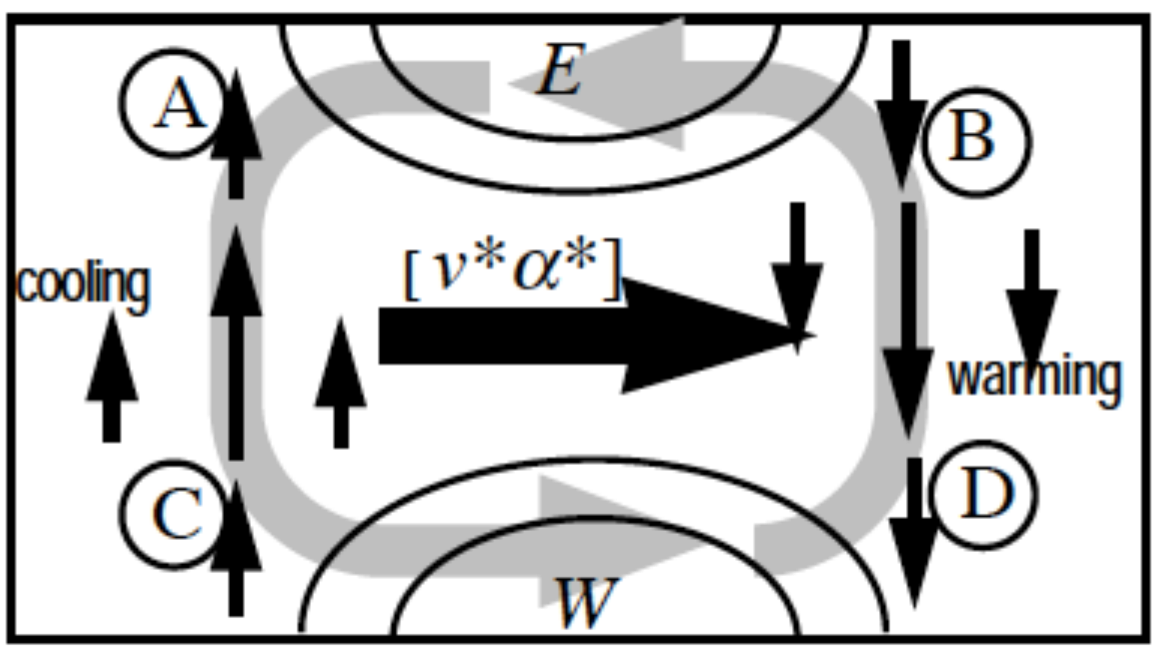


Excuse my ignorance, but I don’t see how the poleward flux of static stability can change the zonal wind field.

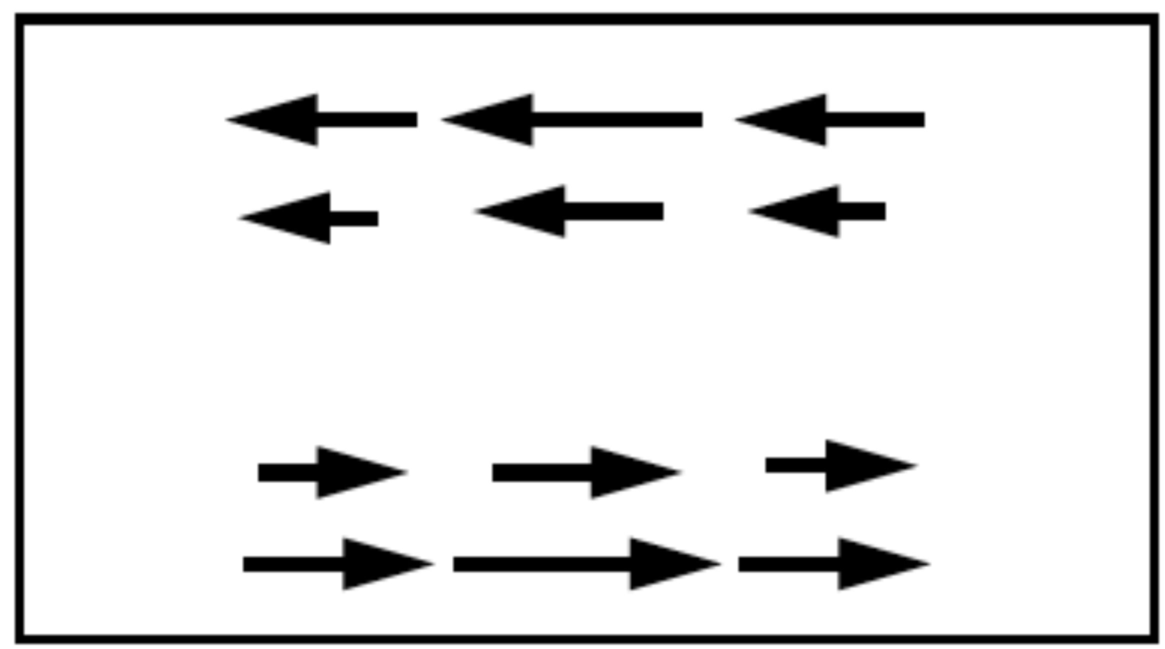
It’s simple if we use the membrane as an analog for potential vorticity

Define $G*$, an imaginary momentum forcing that would change the zonal momentum field at exactly the same rate as the vertical stretching or pinching of the membrane due to the poleward heat fluxes

$$G^* = -f \frac{\partial}{\partial p} \frac{[v^* \alpha^*]}{\sigma}$$



heat flux forcing



equivalent momentum flux forcing

So

$$[q^*v^*] = [\varsigma^*v^*] - f\frac{\partial}{\partial p}\frac{[v^*\alpha^*]}{\sigma}$$

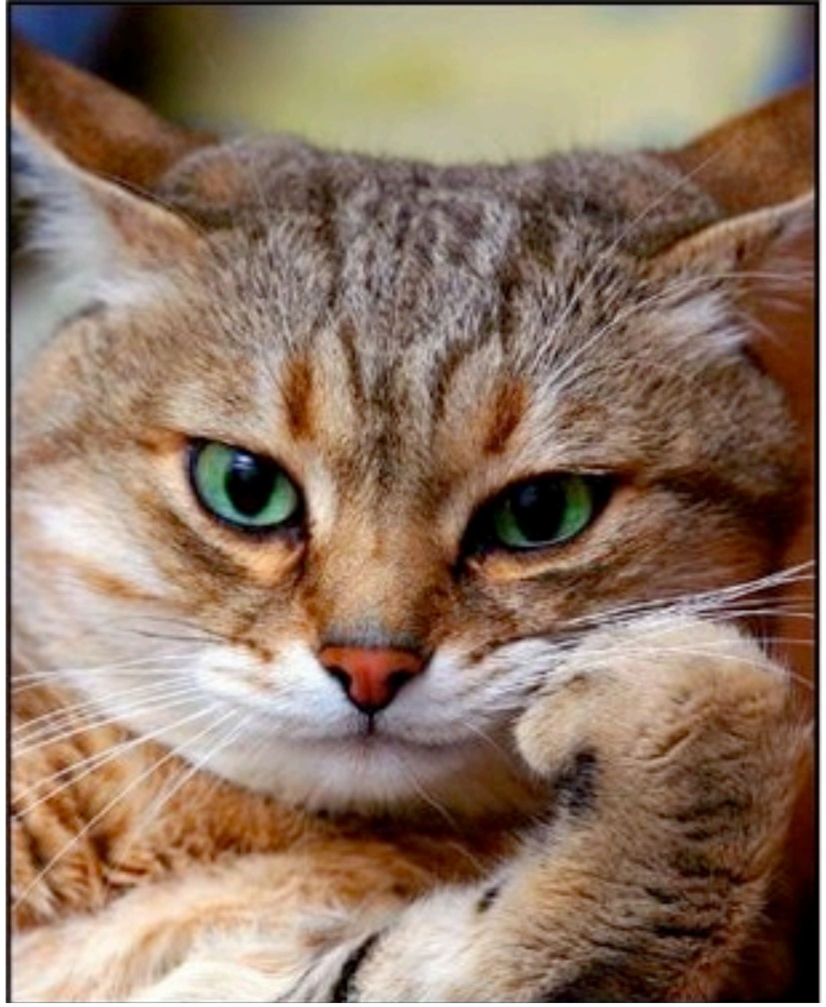
vorticity flux   static stability flux

or

$$[q^*v^*] = G + G^*$$

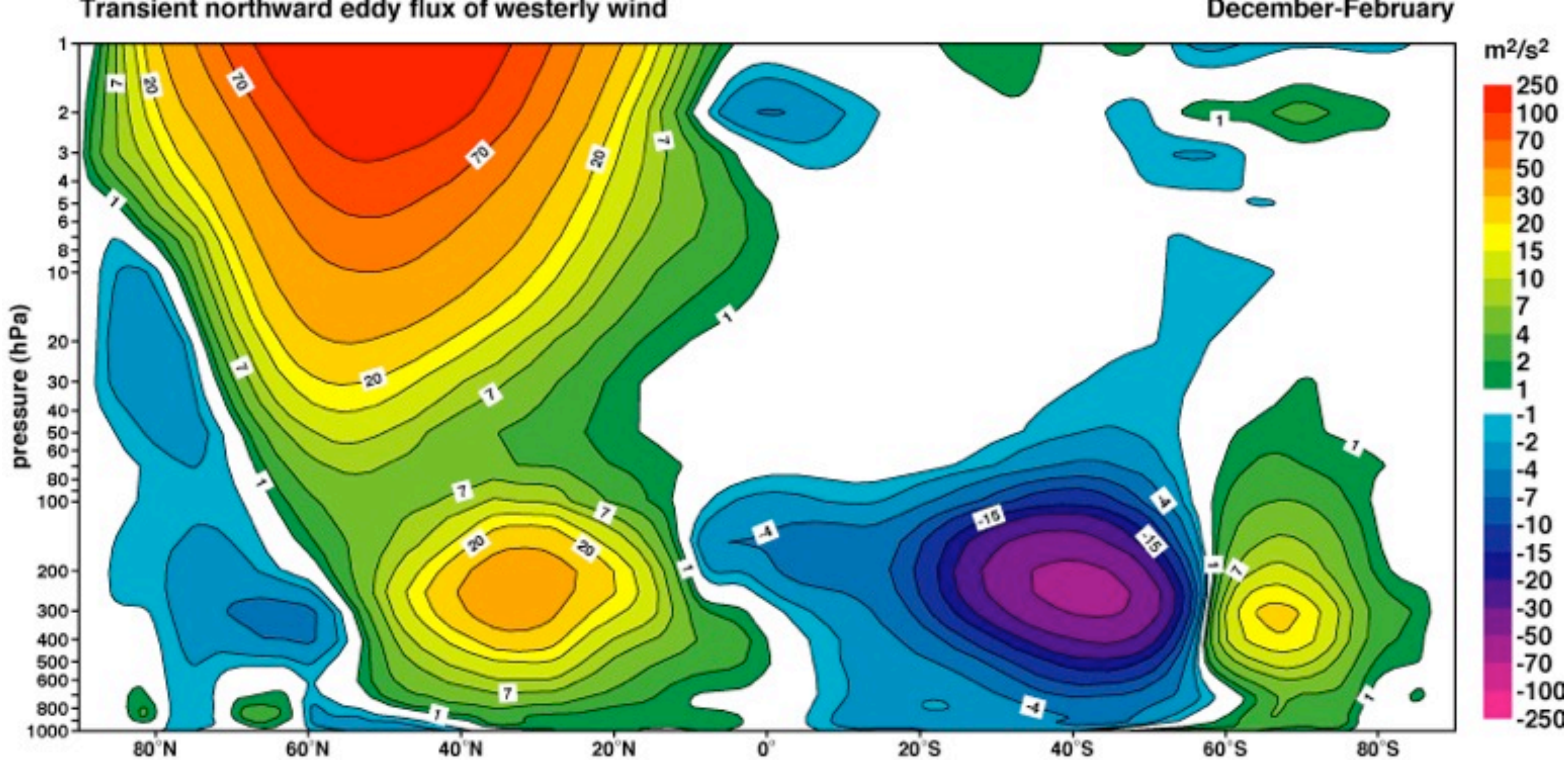
and

$$\left(\frac{du}{dt}\right)_{\substack{eddy\\forced}} = [q^*v^*] = G + G^*$$



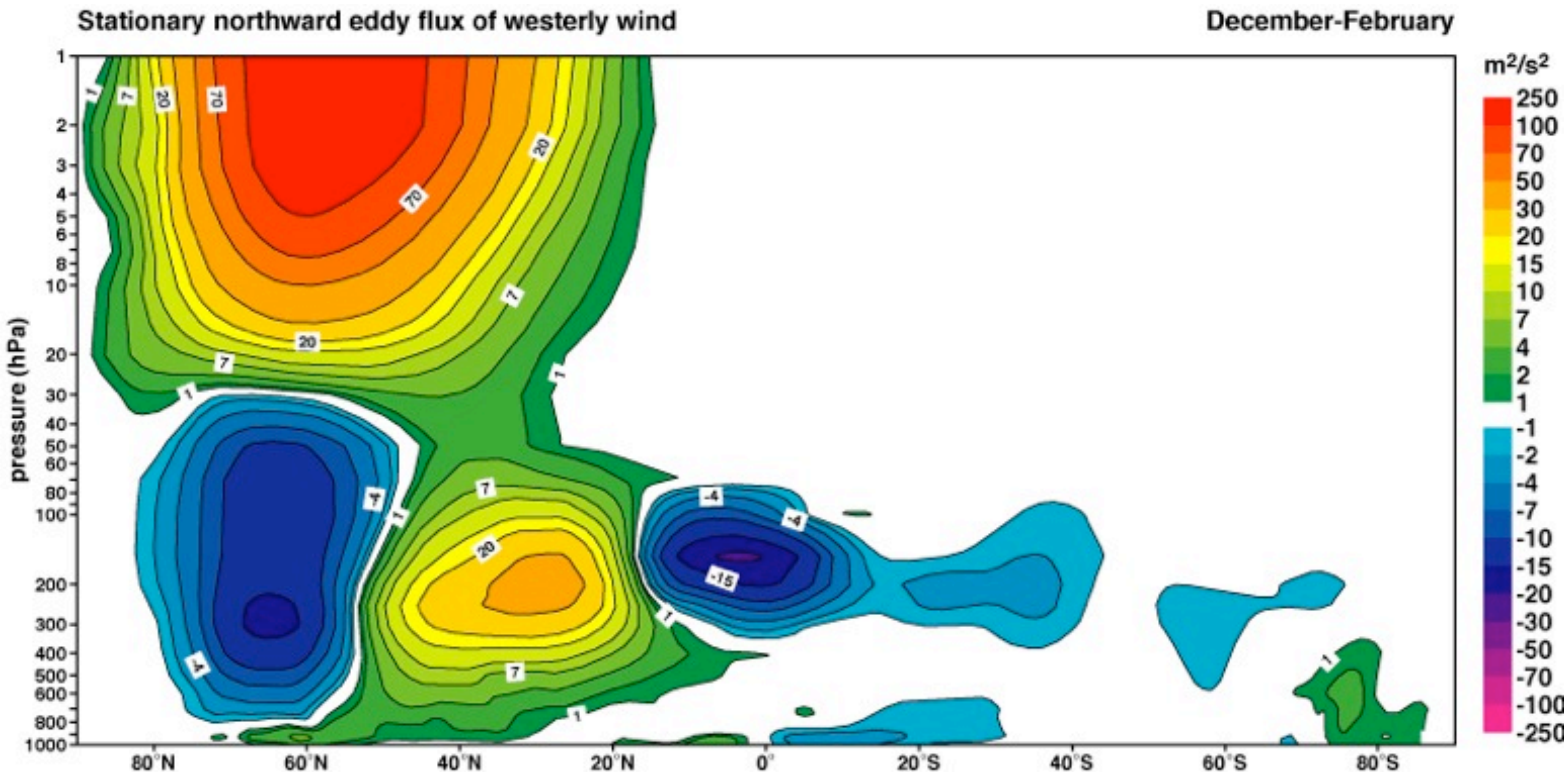
So is that all we need to know?

Well, not quite. The bottom boundary condition is important too. But before we deal with that, let's apply what we've learned in a situation in which the boundary conditions don't matter.

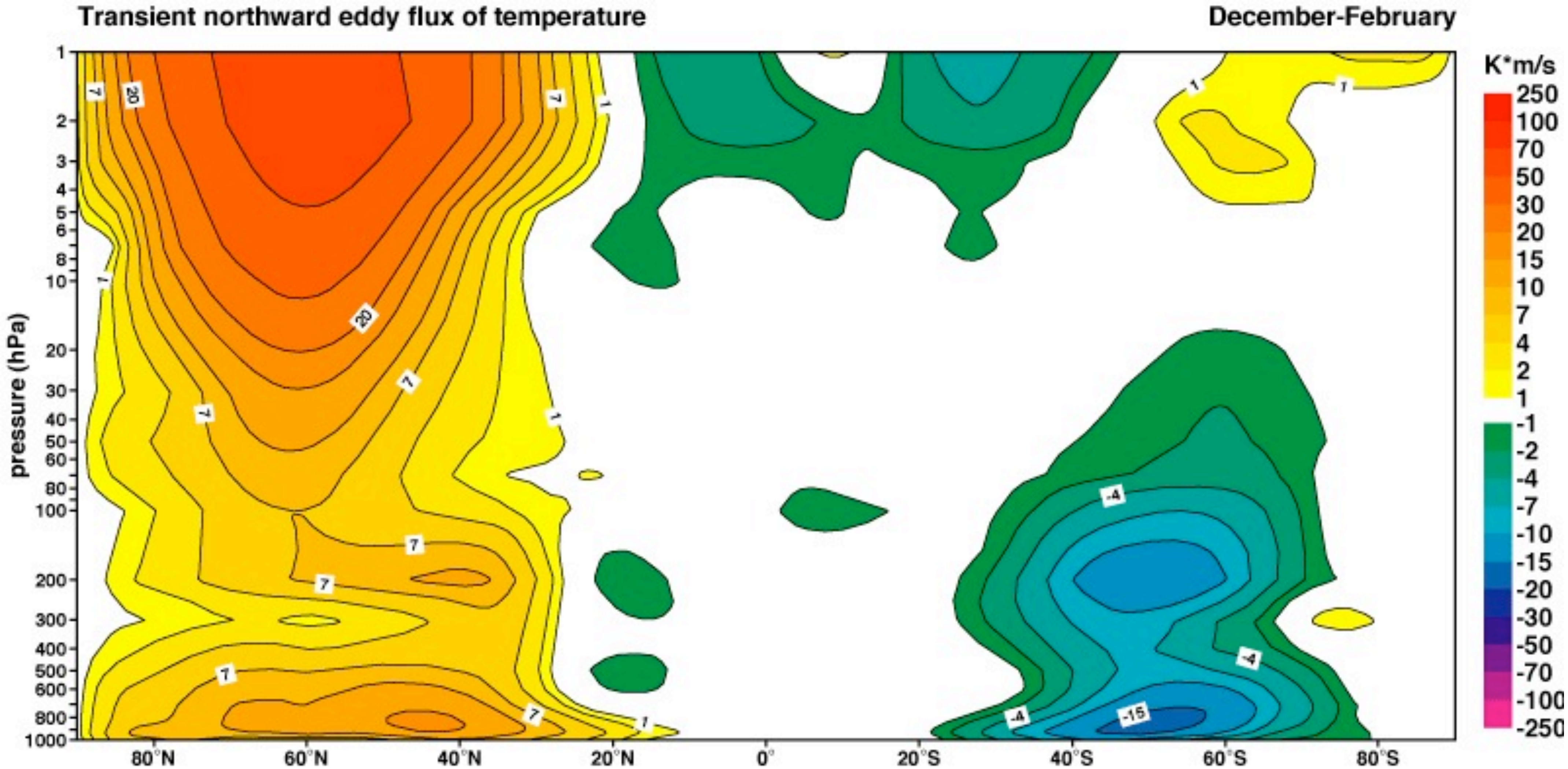


transient eddies

$$[u^*v^*]$$

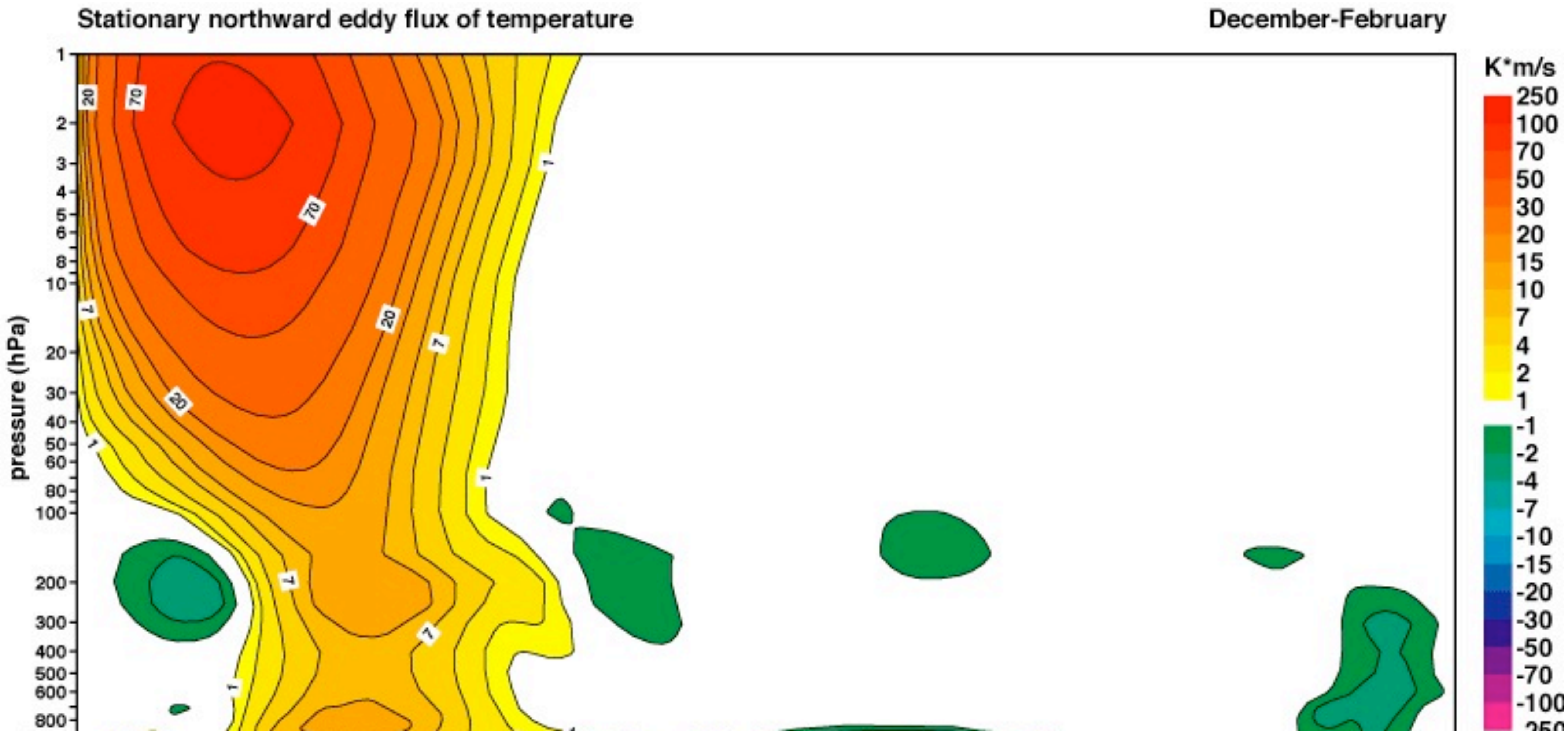


stationary waves

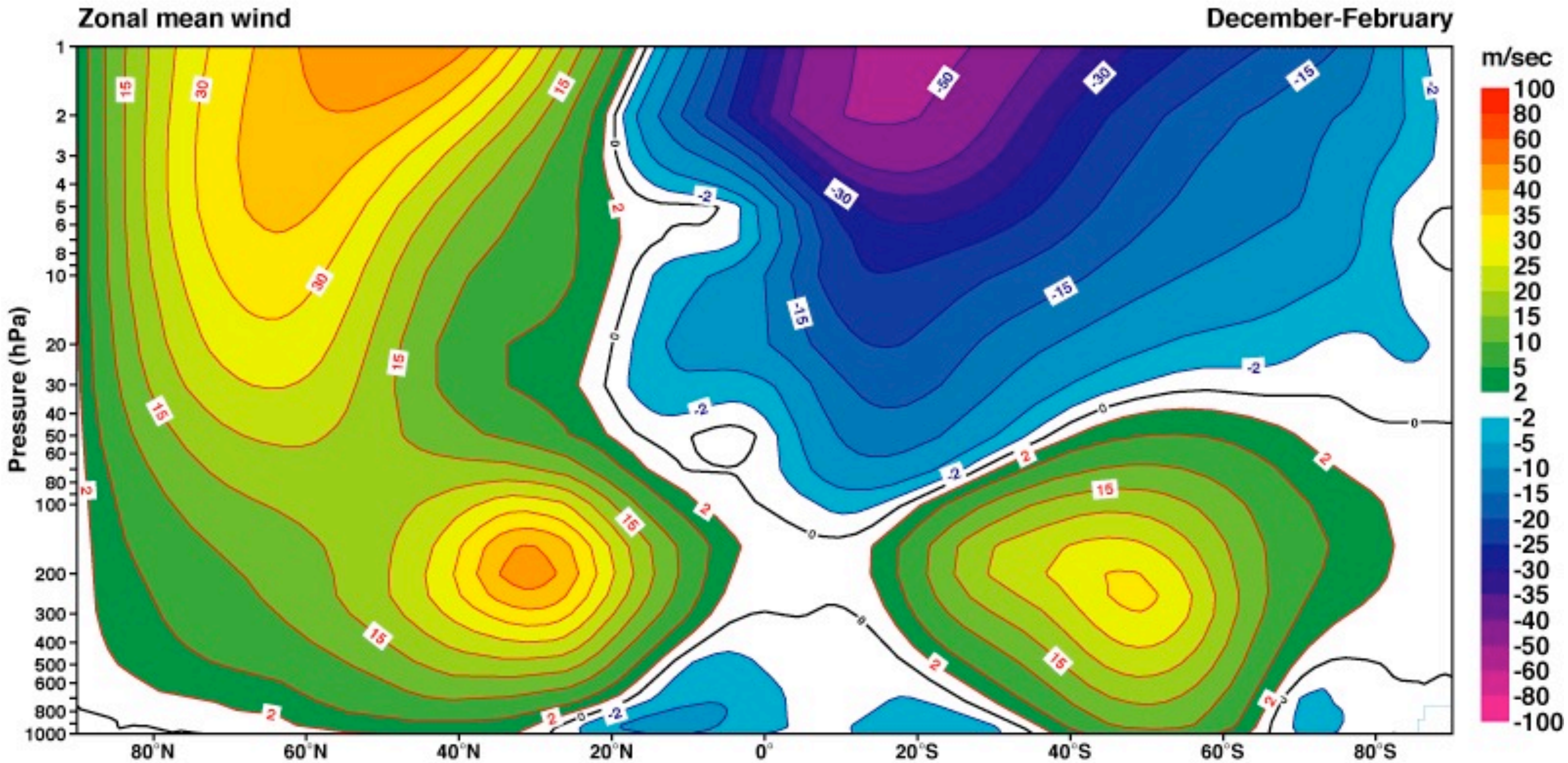


transient
eddies

$$[v^*T^*]$$

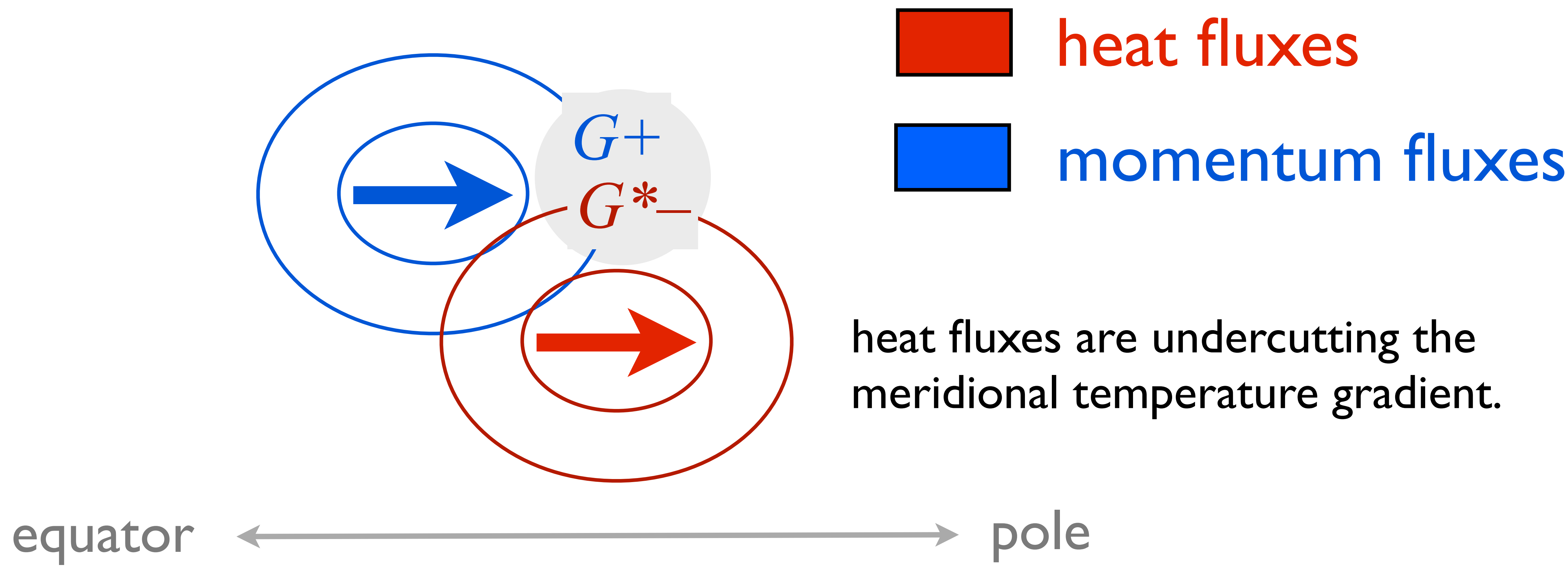


stationary
waves



Zonal mean wind
December-February
Pressure (hPa)
m/sec
80°N
60°N
40°N
20°N
0°
20°S
40°S
60°S
80°S

u



heat fluxes are undercutting the meridional temperature gradient.

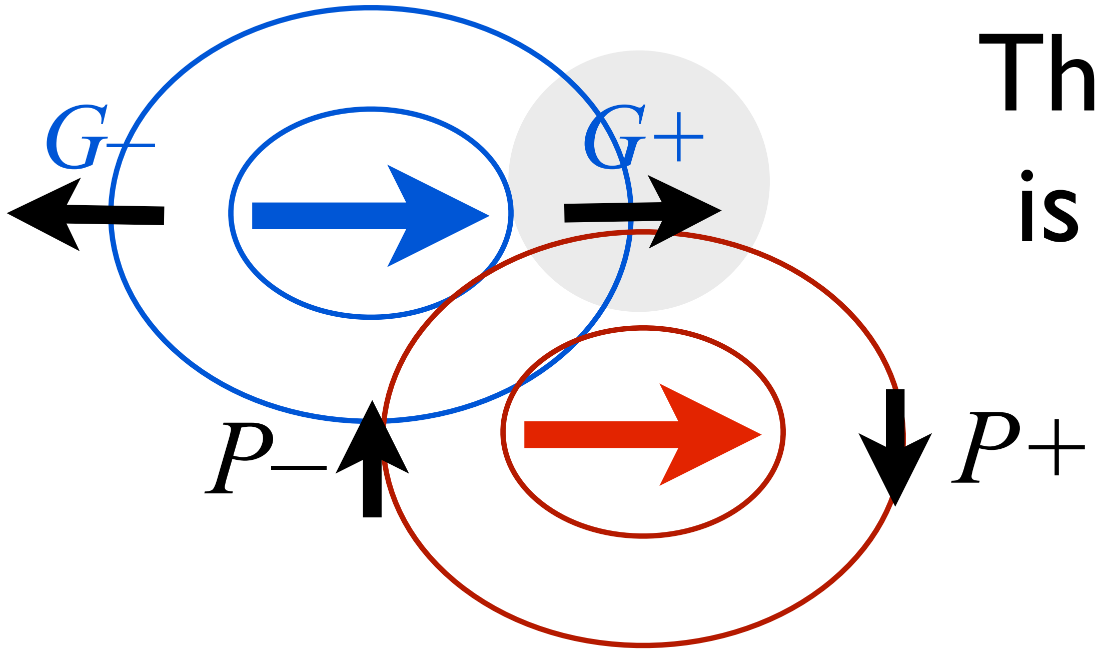
$$G = [\varsigma * v *]$$

$$G^* = -f \frac{\partial}{\partial p} \frac{[v^* \alpha^*]}{\sigma}$$

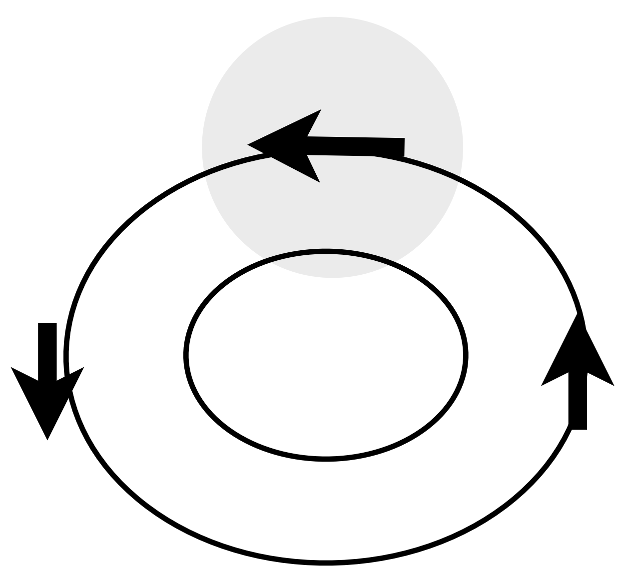
tendency for cancellation

$[q * v *]$ is small

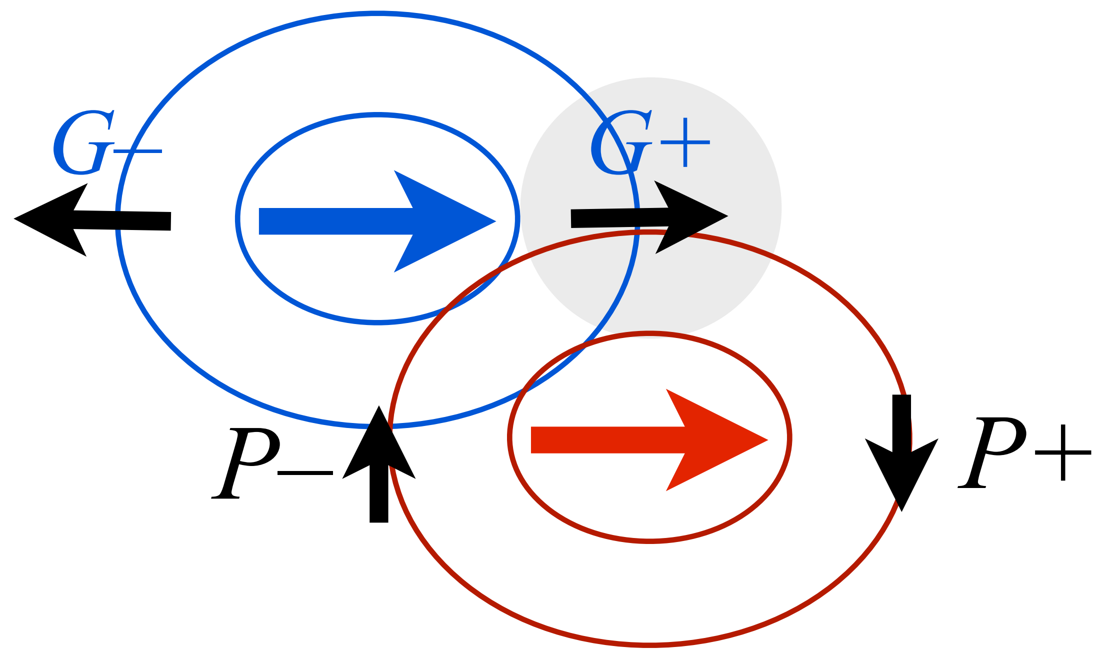
$\frac{du}{dt}$ is small



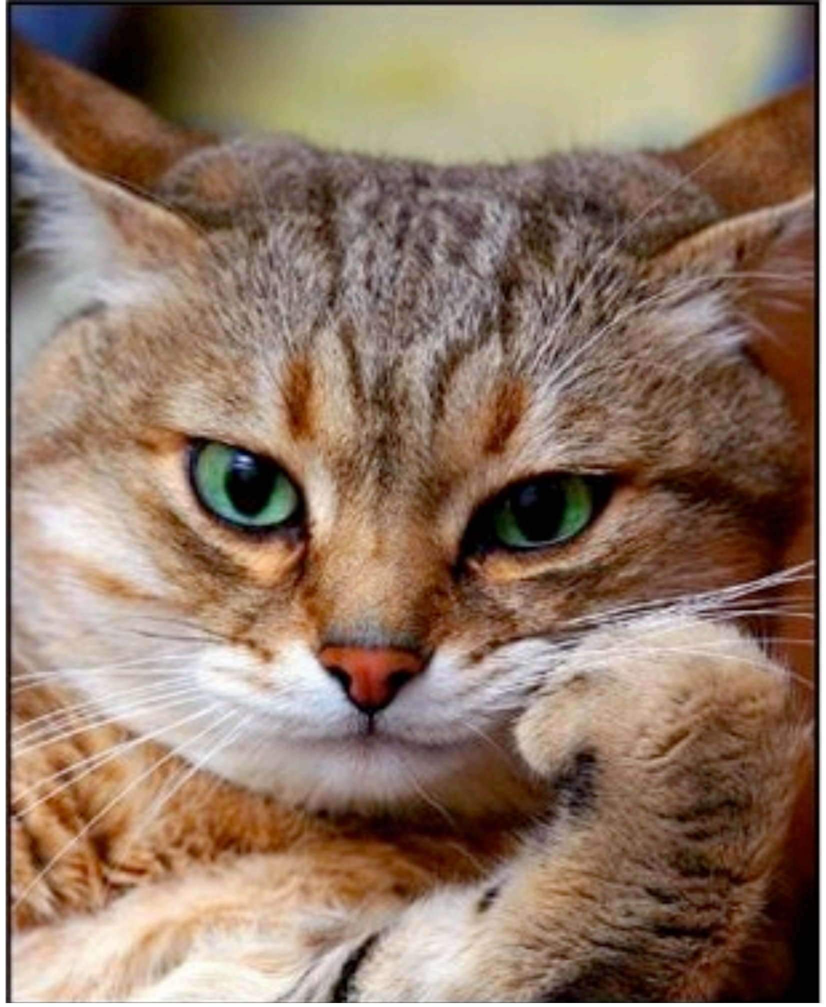
The eddy forcing vector
is mainly nondivergent



and cancelled by
the induced MMC

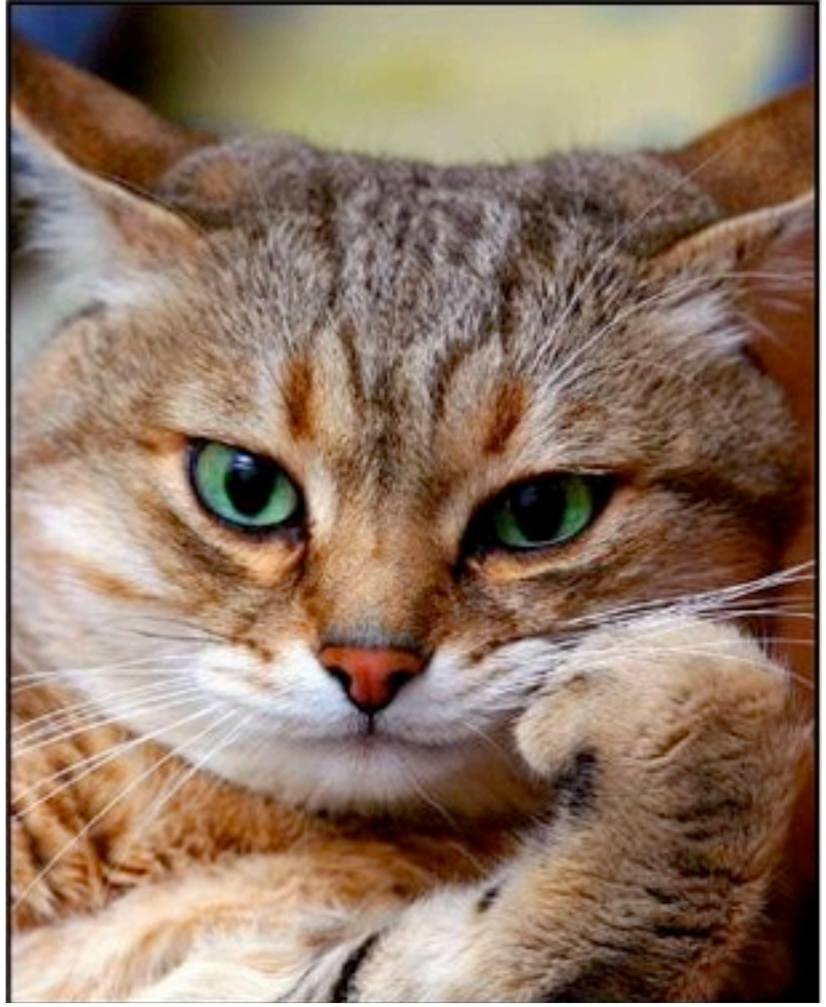


The momentum and heat fluxes are driving the flow out of thermal wind balance, increasing the vertical wind shear while weakening the meridional temperature gradient. The MMC won't let that happen.



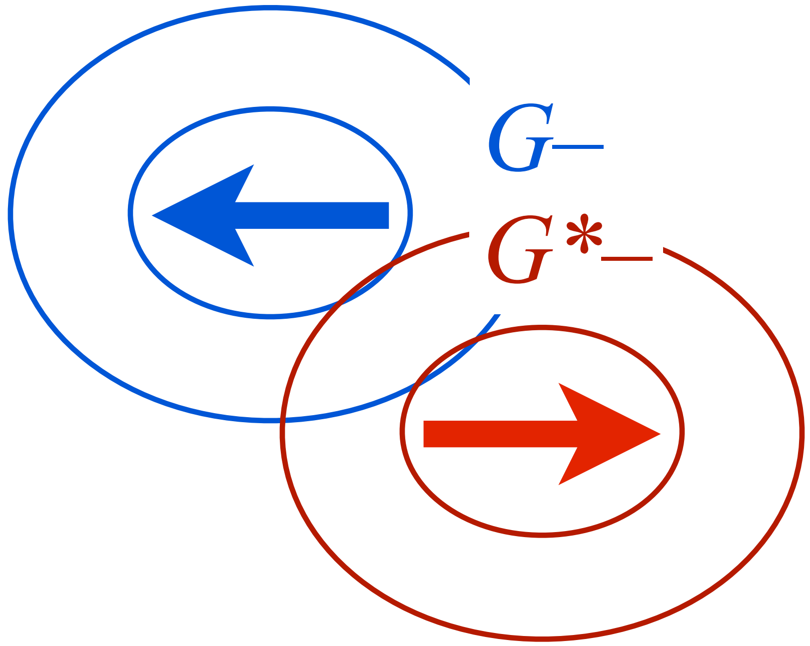
Fluxes running wild and inducing MMC, yet nothing happens to the mean flow?

That’s right. But what if the momentum flux were equatorward rather than poleward?



All hell breaks loose?

Exactly!



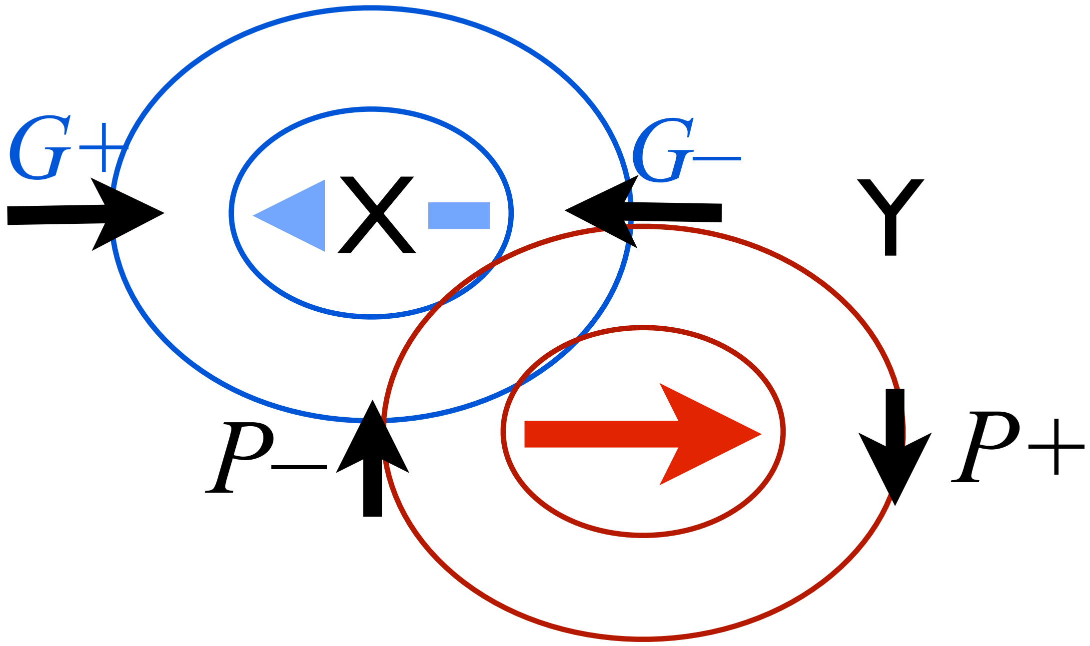
$$G = [\varsigma^* v^*]$$

$$G^* = -f \frac{\partial}{\partial p} \frac{[v^* \alpha^*]}{\sigma}$$

tendency for reenforcement

$[q^* v^*]$ is equatorward

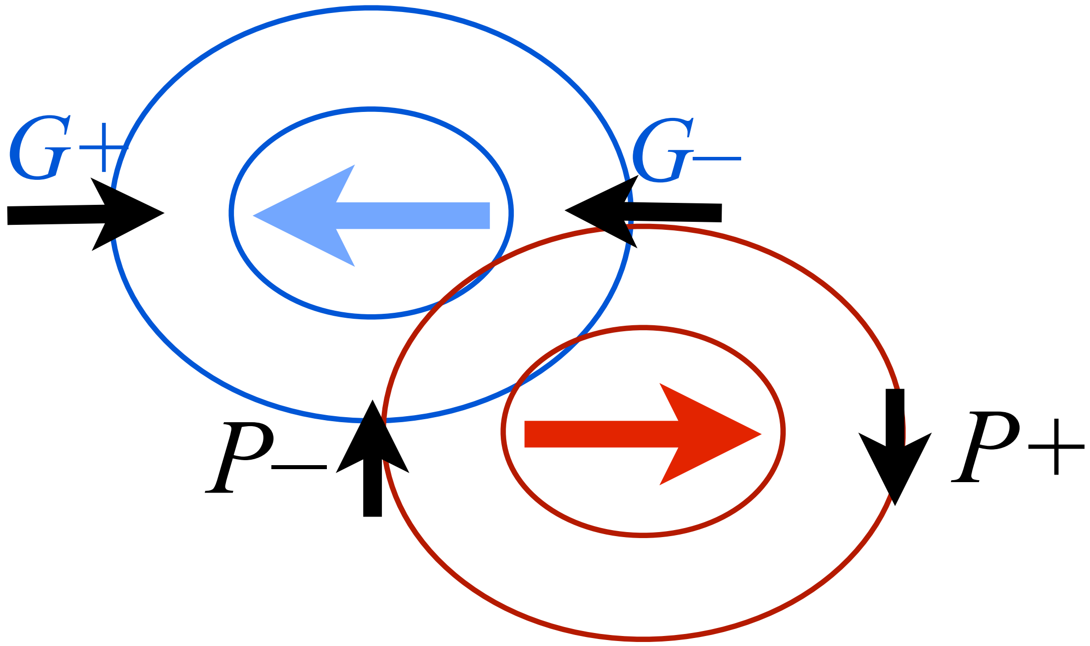
easterly acceleration



in this case the eddy forcing vector is mainly irrotational. The membrane thickens at X and thins at Y

in effect, membrane mass moves equatorward, consistent with $[q^*v^*] = G + G^* < 0.$

The easterly acceleration occurs midway between X and Y, where the forcing is strongest.



in this case there is little or no eddy-induced MMC because the forcing field is nearly irrotational

The zonal flow isn't being forced away from thermal wind balance. The vertical shear and the meridional temperature gradient are both being forced to decrease.

First you prescribe a distribution of eddy fluxes supposedly based on climatology. Then you reverse the sign of the momentum fluxes. Reversing fluxes.. how could this be? Does it happen in nature?

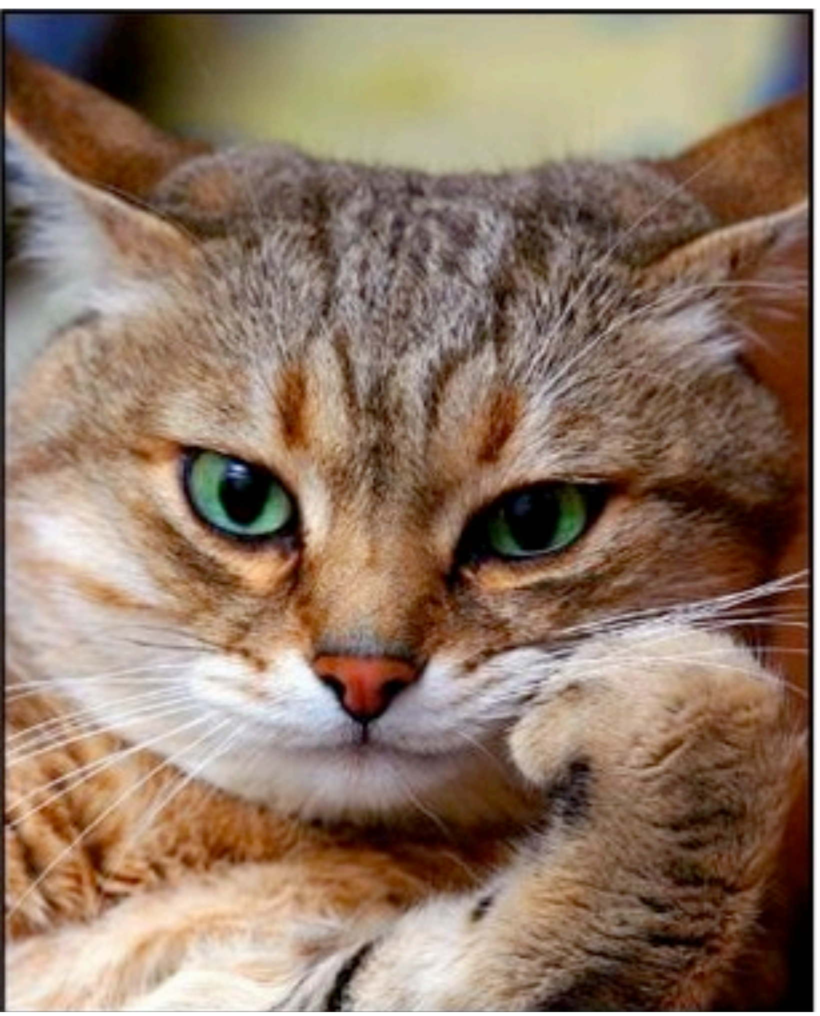
Believe it or not, it does,
every winter in the stratosphere!

For week after week, the patterns resemble the climatology. The so-called *polar night jet* is even stronger than in the climatology (~50 m/s at 10 hPa) and temperatures over the polar cap region approach –80°C, the threshold for the formation of polar stratospheric clouds. The poleward eddy heat fluxes are strong, but the tendencies that they induce are almost exactly cancelled by the MMC.

Then every so often (on average once or twice per winter) the momentum fluxes reverse and within a few days, the polar vortex moves off the pole and weakens while temperatures over the polar cap region rise by 50°C or more.

They're awesome!

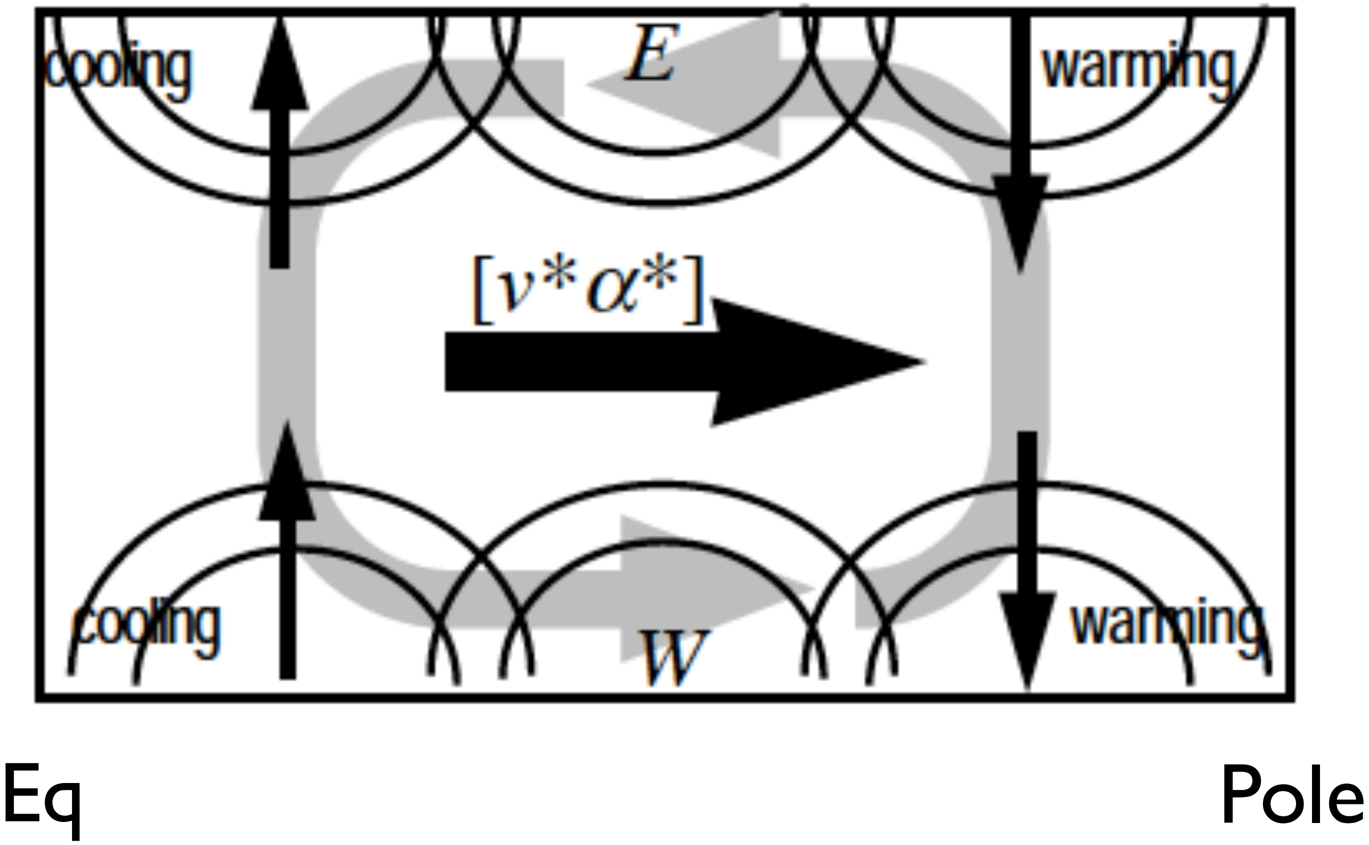
Especially the animations showing the potential vorticity field.



But they're no longer on the web site

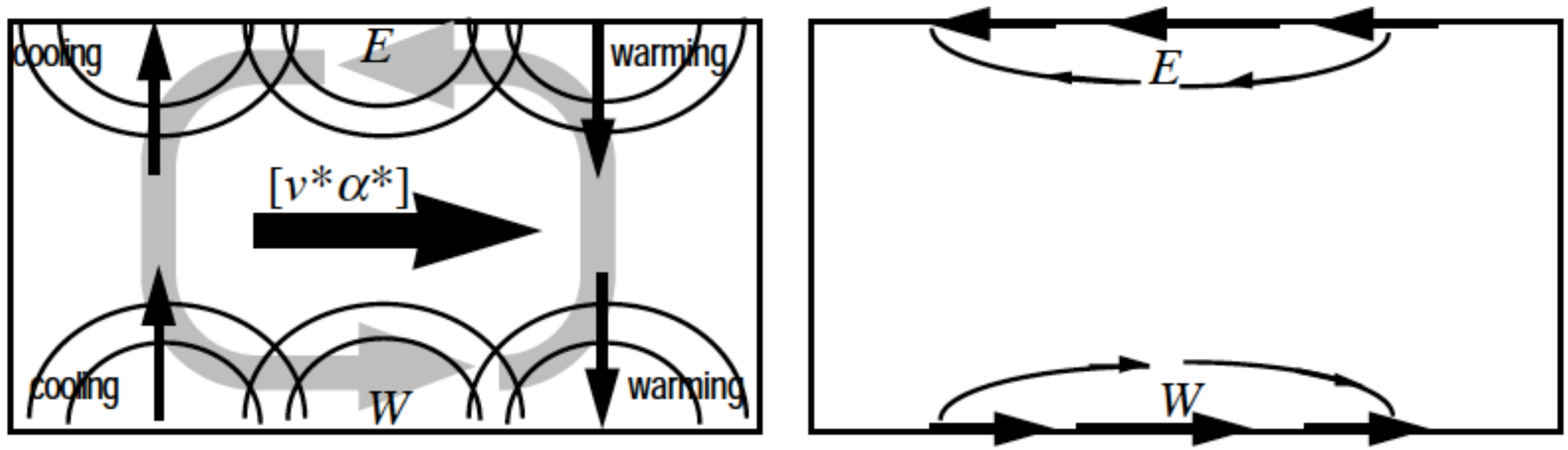
I’m intrigued with the bottom boundary condition. Can you tell us more about that?

It’s those heat fluxes that we talked about earlier



Imagine the situation depicted above.
No momentum fluxes, friction, or diabatic heating.
Heat fluxes not varying with height.

The heat fluxes extend all the way to the top and bottom boundaries where the vertical component of the MMC has to vanish.

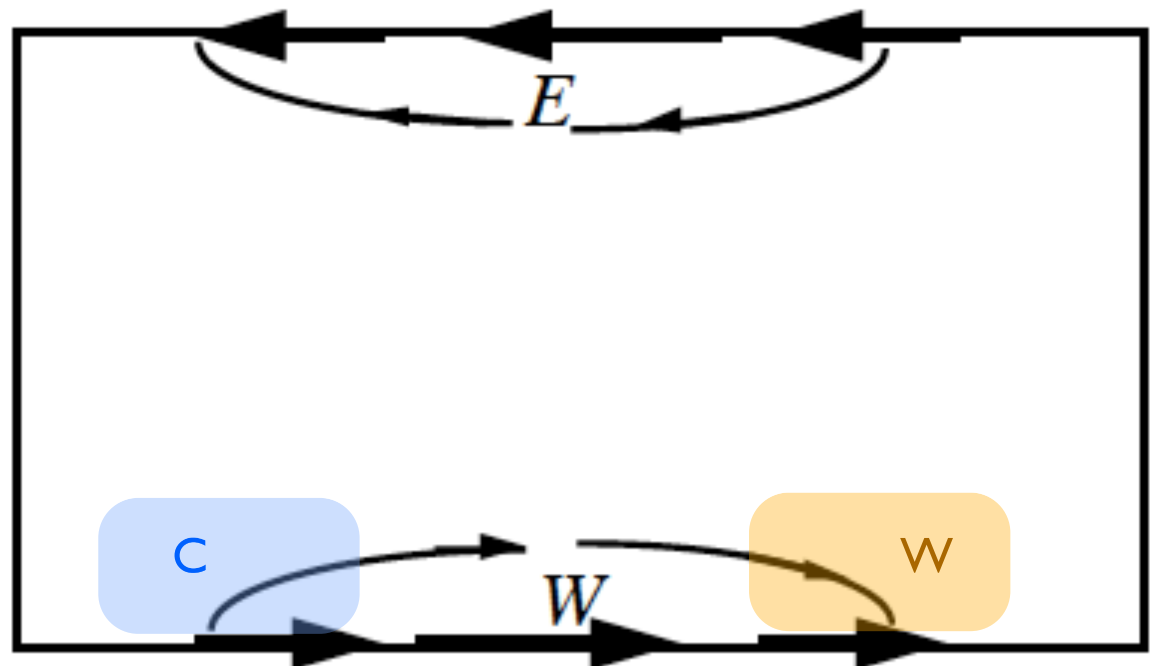


Eq Pole

To satisfy this constraint it is necessary to invoke a boundary-forced component of the solution.

We allow potential vorticity (membrane mass) to accumulate in reservoirs the top and boundaries as prescribed by the forcing vectors (left).

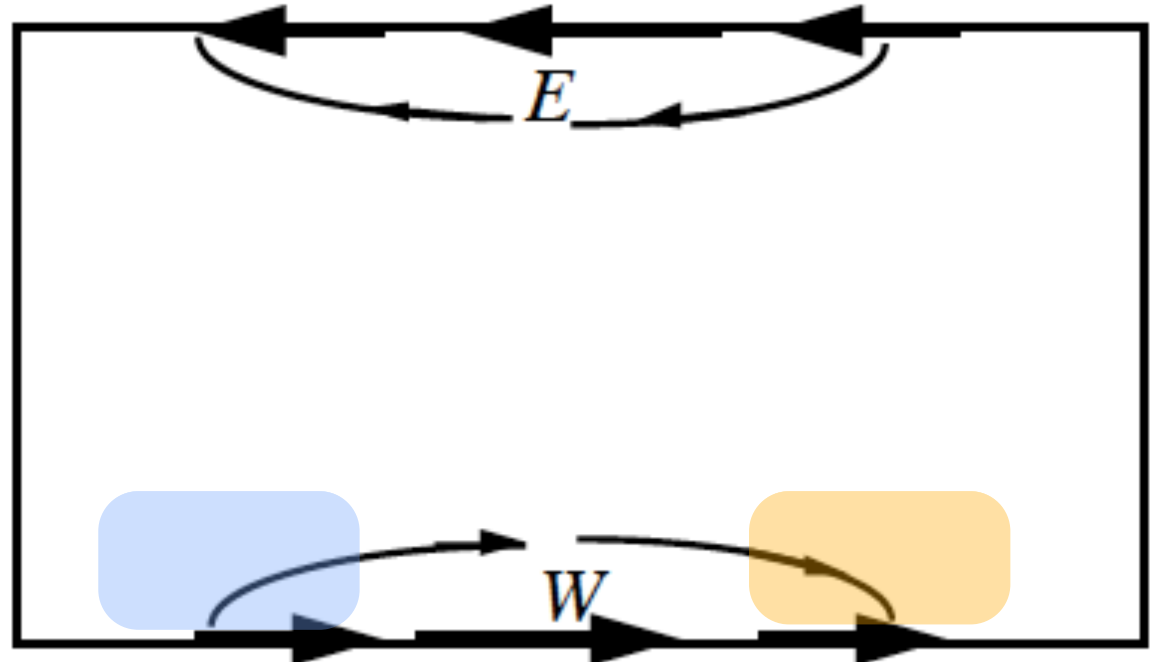
This is equivalent to moving membrane mass from left to right along the bottom boundary (right).



Because of the inherent stiffness of the membrane, mass must also move meridionally within some finite depth, as determined by the modulus of elasticity and there are associated vertical displacements of the membrane.

The poleward flux of potential vorticity induces a westerly acceleration on and just above the bottom boundary, as indicated.

The associated vertical velocities weaken the meridional temperature gradient just above the bottom boundary



Bearing in mind that the induced zonal wind and temperature tendencies are both weakening with distance from the top and bottom boundaries, the induced changes are consistent with geostrophy.

Above the bottom boundary, $\partial u / \partial z$ and $\partial T / \partial y$ are both weakened by the poleward heat flux on the boundary.

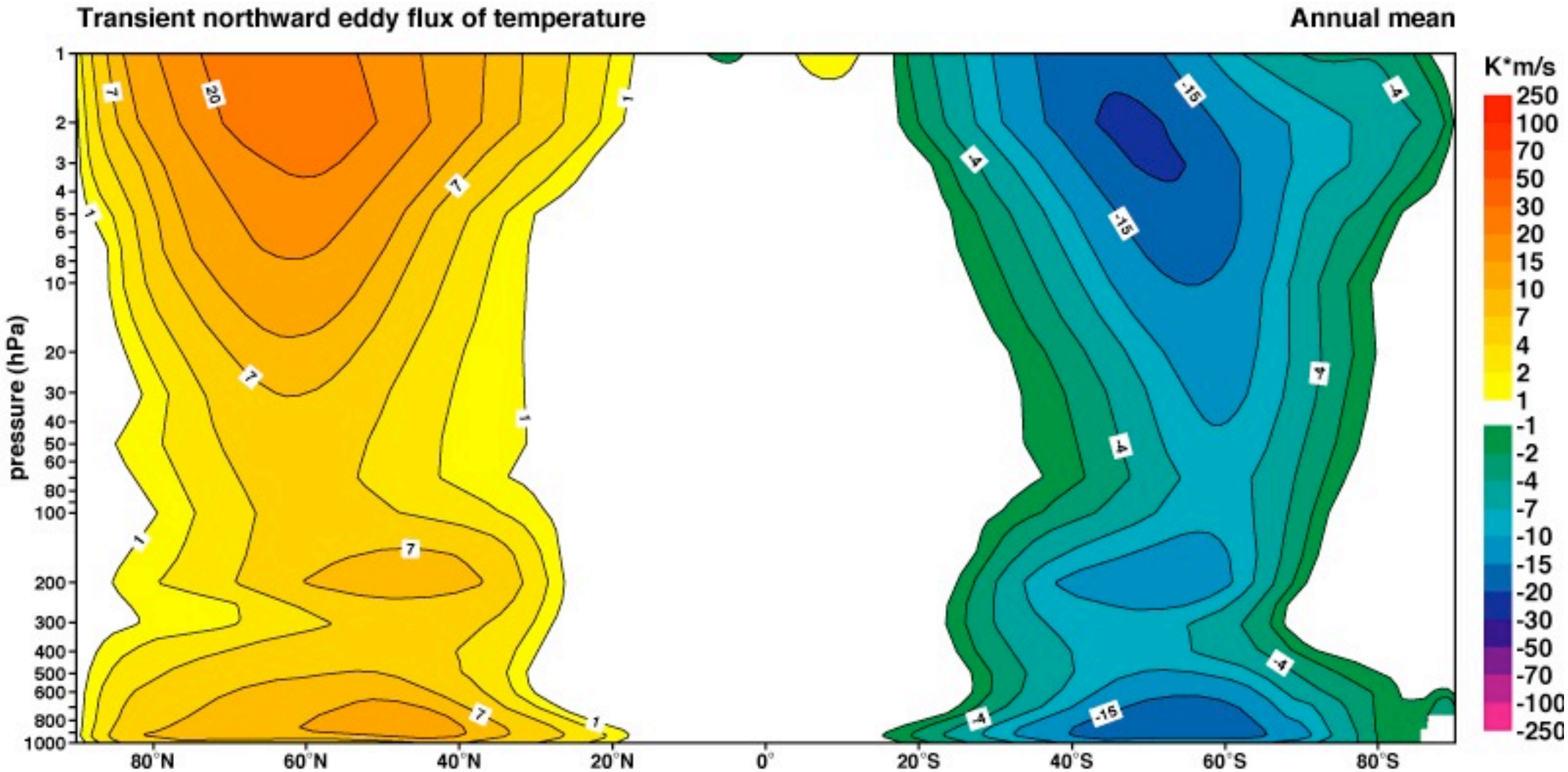
Smells like barotropification.

Exactly! The poleward heat flux at the bottom boundary, in combination with the heat fluxes in the interior and their induced MMC, weaken the meridional temperature gradient and vertical wind shear from the ground up.

I think I’ve smelled this before.

You have. The geostrophic temperature advection at the bottom boundary plays an equivalent role in the quasi-geostrophic system and it is central to the theory of baroclinic instability.

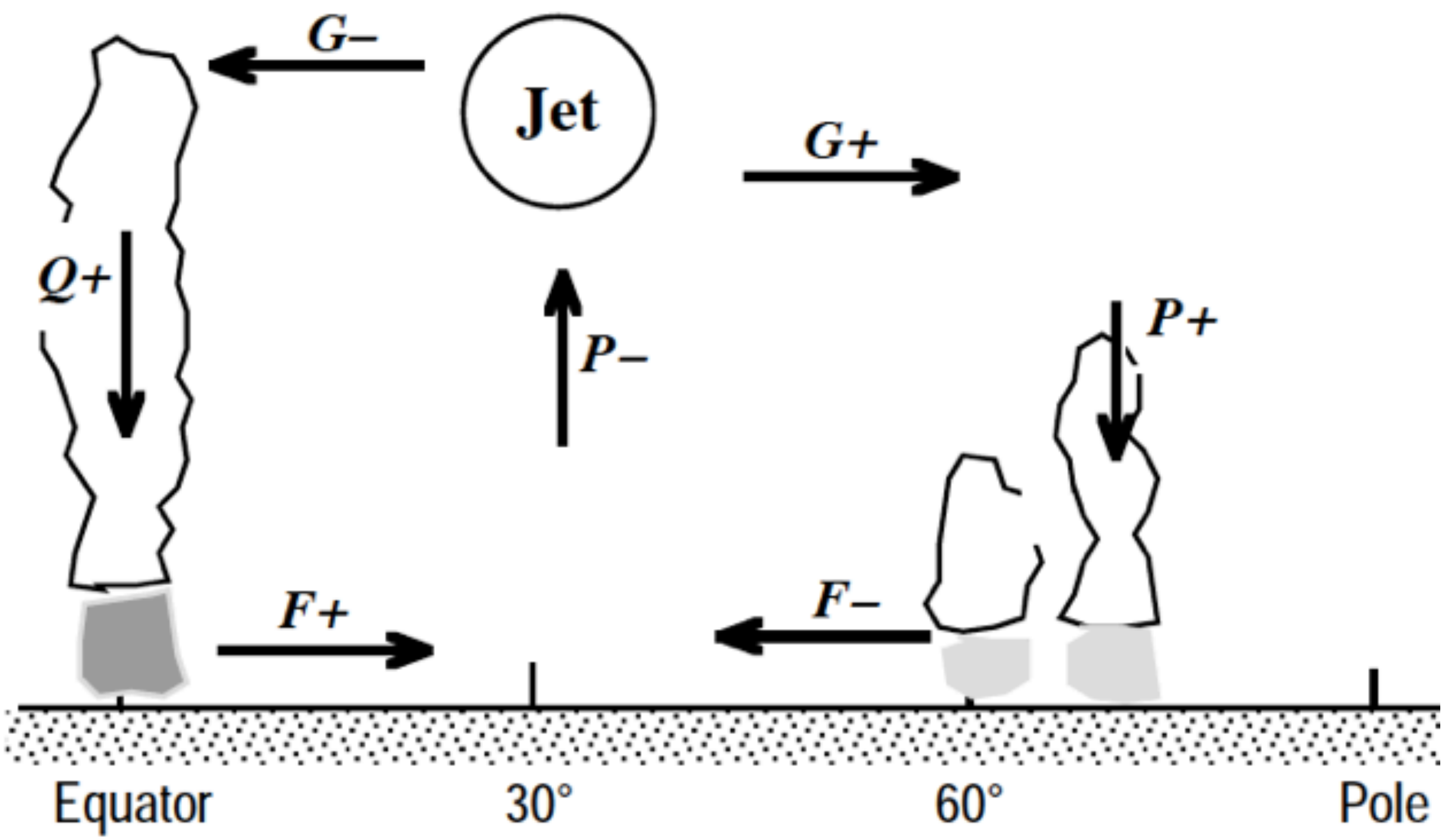
Do the heat fluxes really extend down to the bottom boundary?



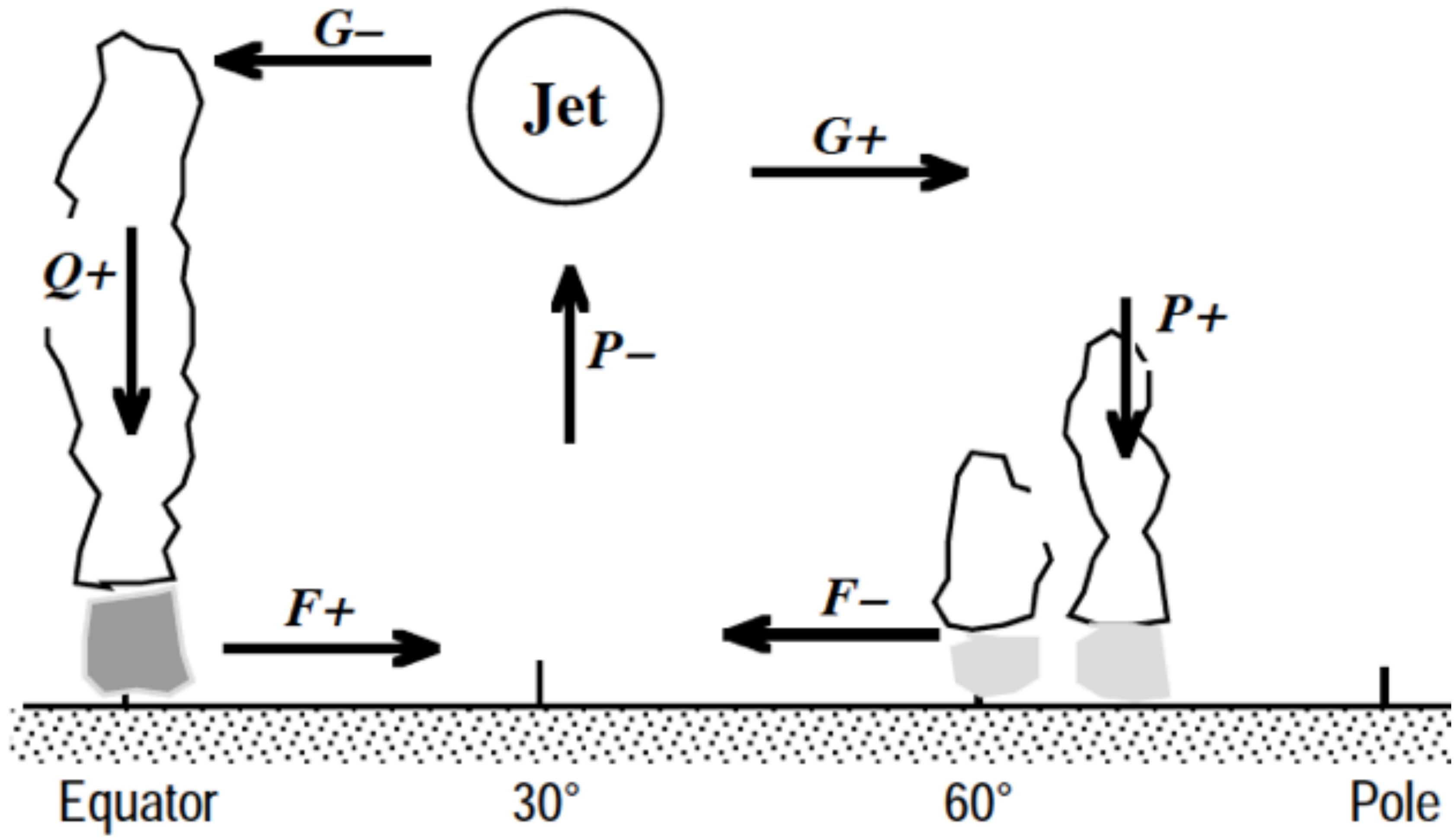
Yes, they do. Note the maximum at the top of the boundary layer.

How do you expect us to remember all this stuff?

Let's do a few exercises to help it sink in.



Here's the climatology. Now what would happen if we abruptly double the strength of $[u^*v^*]$?



OK. Now let's do the same with $[v^*T^*]$? What happens?

Note how the MMC always act to oppose the forcing.

Kind of like the way I behave in class.

Yet in the very act of opposing the forcing, they change the field that isn't being forced in the sense that is geostrophically consistent with the forcing.

In both the above examples, the system does change in response to the forcing but, thanks to the MMC, it does so in a geostrophically consistent way.

These conclusions apply equally well regardless of with the forcing is due to eddy fluxes, diabatic heating, or friction.