

APPENDIX II

THE ZONAL MOMENTUM BALANCE

The equation that governs the local time rate of change of zonal wind can be written in the form

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \omega \frac{\partial u}{\partial p} + \frac{uv \tan \phi}{R_E} - \frac{\partial \Phi}{\partial x} + fv + F_x \quad (\text{A } 2.1)$$

A complete derivation of this equation is given in Holton (1972) p. 21-28*.

The advective terms can be rewritten in the form

$$-u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \omega \frac{\partial u}{\partial p} = -\frac{\partial}{\partial x} u^2 - \frac{1}{\cos \phi} \frac{\partial}{\partial y} uv \cos \phi - \frac{\partial}{\partial p} u\omega + u \left(\frac{\partial u}{\partial x} + \frac{1}{\cos \phi} \frac{\partial}{\partial y} v \cos \phi + \frac{\partial \omega}{\partial p} \right)$$

where the term in parentheses vanishes because of the continuity of mass.

Substituting back into (A. 2.1) and making use of the identity

$$\frac{1}{\cos^2 \phi} \frac{\partial}{\partial y} (uv \cos^2 \phi) = \frac{1}{\cos \phi} \frac{\partial}{\partial y} uv \cos \phi - \frac{uv \tan \phi}{R_E}$$

we obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} u^2 - \frac{1}{\cos^2 \phi} \frac{\partial}{\partial y} uv \cos^2 \phi - \frac{\partial}{\partial p} u\omega - \frac{\partial \Phi}{\partial x} + fv + F_x \quad (\text{A } 2.2)$$

When we zonally average, the terms $-\partial/\partial x(u^2)$ and $-\partial\Phi/\partial x$ drop out because of the identity

$$[\partial () / \partial x] = \frac{1}{R_E \cos \phi} \int_0^L \frac{\partial ()}{\partial x} dx = () \Big|_0^L = 0 \quad (\text{A } 2.3)$$

*Here the equation has been written in pressure coordinates and the small terms $u\omega/R_E$ and $2\Omega u \cos \phi$ have been neglected. It is easily shown that these terms are at least 2 orders of magnitude smaller than the corresponding terms in v .

Next we expand the $[uv]$ and $[u\omega]$ terms, making use of (A. 1.11), to obtain

$$\begin{aligned} \frac{\partial[u]}{\partial t} = & \frac{-1}{\cos^2\phi} \frac{\partial}{\partial y} [u][v] \cos^2\phi - \frac{\partial}{\partial p} [u][\omega] + f[v] + [F_x] \\ & - \frac{1}{\cos^2\phi} \frac{\partial}{\partial y} [u^*v^*] \cos^2\phi - \frac{\partial}{\partial p} [u^*\omega^*] \end{aligned} \quad (\text{A } 2.4)$$

Then we expand the mean meridional motion terms in the form

$$\frac{1}{\cos^2\phi} \frac{\partial}{\partial y} [u][v] \cos^2\phi = \frac{[u]}{\cos\phi} \frac{\partial}{\partial y} [v] \cos\phi + \frac{[v]}{\cos\phi} \frac{\partial}{\partial y} [u] \cos\phi$$

and

$$\frac{\partial}{\partial p} [u][\omega] = [u] \frac{\partial[\omega]}{\partial p} + [\omega] \frac{\partial[u]}{\partial p}$$

Substituting back into (A 2.4) and making use of the zonally averaged continuity equation in spherical coordinates

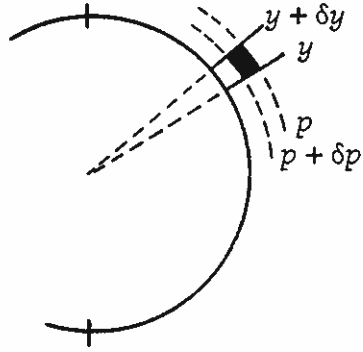
$$\frac{1}{\cos\phi} \frac{\partial}{\partial y} [v] \cos\phi + \frac{\partial[\omega]}{\partial p} = 0$$

we obtain, after some minor rearranging,

$$\begin{aligned} \frac{\partial[u]}{\partial t} = & [v] \left(f - \frac{1}{\cos\phi} \frac{\partial}{\partial y} [u] \cos\phi \right) - [\omega] \frac{\partial[u]}{\partial p} \\ & - \frac{1}{\cos^2\phi} \frac{\partial}{\partial y} [u^*v^*] \cos^2\phi - \frac{\partial}{\partial p} [u^*\omega^*] + [F_x] \end{aligned} \quad (\text{A } 2.5)$$

As an alternative method of deriving (A 2.5) we can start with the equation governing the angular momentum of a fixed, zonally symmetric annulus, bounded by latitudinal "walls" at y and $y + \delta y$ and pressure levels p and $p + \delta p$, as

shown in the accompanying figure. The only processes capable of changing the



integrated angular momentum within the annulus are advection across the boundaries of the annulus and frictional torques acting within the annulus. Such torques will be assumed to be small unless the annulus is contiguous with the earth's surface. The net advection of angular

momentum across the latitudinal walls is given by

$$\int \int M v \, dx \, dp \Big|_y - \int \int M v \, dx \, dp \Big|_{y + \delta y}$$

where the zonal integration is carried out around a complete latitude circle and the vertical integration is carried out from level p down to $p + \delta p$. Expanding Mv in a Taylor series expansion in y , and keeping only the linear term, the above expression reduces to $-\left\{ \int \int (\partial Mv / \partial y) \, dx \, dp \right\} \delta y$, which is an accurate representation, provided that δy is sufficiently small. Furthermore, if δp is sufficiently small, this expression can be vertically integrated to obtain $-\left\{ \int \partial Mv / \partial y \, dx \right\} \delta y \delta p$, or, using (A 1.2), $-2\pi R_E \delta y \delta p \frac{\partial}{\partial y} [Mv] \cos \phi$. In a similar manner, the net vertical advection reduces to $-2\pi R_E \cos \phi \delta y \delta p \partial [M\omega] / \partial p$. [In writing the vertical advection term in pressure coordinates we are implicitly neglecting the vertical convergence or divergence associated with the variation in δy with height. It is this effect which leads to the small $(u\omega \tan \phi) / R_E$ term which is neglected in (A 2.1) (see footnote, p. A7).

Now let us proceed by writing the equation for the angular momentum balance

$$\begin{aligned} \frac{\partial}{\partial t} \iiint M dx dy dp &= -2\pi R_E \delta y \delta p \left\{ \frac{\partial}{\partial y} [Mv] \cos\phi + \cos\phi \frac{\partial}{\partial p} [M\omega] \right\} \\ &+ \iiint F_x R_E \cos\phi dx dy dp \end{aligned}$$

Integrating the remaining terms over y and p , zonally averaging them, and dividing through by $2\pi R_E \cos\phi \delta y \delta p$ we obtain

$$\frac{\partial [M]}{\partial t} = -\frac{1}{\cos\phi} \frac{\partial}{\partial y} [Mv] \cos\phi - \frac{\partial}{\partial p} [M\omega] + [F_x] R_E \cos\phi \quad (\text{A } 2.6)$$

Repeating the same sequence of operations that we applied in transforming (A 2.4) into (A 2.5), we obtain

$$\begin{aligned} \frac{\partial [M]}{\partial t} &= -[v] \frac{\partial [M]}{\partial y} - [\omega] \frac{\partial [M]}{\partial p} + [F_x] R_E \cos\phi \\ &- \frac{1}{\cos\phi} \frac{\partial}{\partial y} [M^* v^*] \cos\phi - \frac{\partial}{\partial p} [M^* \omega^*] \end{aligned} \quad (\text{A } 2.7)$$

This expression can be rewritten in the form

$$\frac{D[M]}{Dt} = -\frac{1}{\cos\phi} \frac{\partial}{\partial y} [M^* v^*] \cos\phi - \frac{\partial}{\partial p} [M^* \omega^*] + [F_x] R_E \cos\phi \quad (\text{A } 2.8)$$

Where D/Dt represents the time rate of change following a zonally symmetric ring as it moves with the mean meridional circulations. Such a ring can experience changes in angular momentum only through convergences (or divergences) of eddy flux or through friction

Now when we substitute

$$[M] = \Omega R_E^2 \cos^2\phi + [u] R_E \cos\phi$$

$$\frac{\partial [M]}{\partial t} = R_E \cos\phi \frac{\partial [u]}{\partial t}$$

$$\frac{\partial [M]}{\partial y} = -R_E \cos\phi \left\{ f - \frac{1}{\cos\phi} \frac{\partial}{\partial y} [u] \cos\phi \right\} \quad \text{and}$$

$$\frac{\partial [M]}{\partial p} = R_E \cos\phi \frac{\partial [u]}{\partial p}^\dagger$$

into (A 2.7), and divide through by $R_E \cos\phi$ we obtain an expression identical to (A 2.5). A distinct advantage of this second method of deriving (A 2.5) is that it suggests a clear physical interpretation of the various terms.

[†]A small approximation is involved here in neglecting the vertical derivative of the earth's angular momentum. This approximation is equivalent to neglecting the term $2\Omega w \cos\phi$ in (A 2.1). See also the footnote on p. 47.