Lax-Wendroff method (D2.5.2)

If $\psi_t + c\psi_x = 0$, we can find a ~~two-level~~ single stage, 2nd order accurate method as follows:

$$\psi(x_j, t^{n+1}) = \psi(x_j, t^n) + \Delta t\, \psi_t(x_j, t^n) + \frac{\Delta t^2}{2}\psi_{tt}(x_j, t^n) + \frac{\Delta t^3}{6}\psi_{ttt}(x_j, t_n)$$

The L-W method approximates each of the first three terms with error $O(\Delta t^3)$ by using $\psi_t = -c\psi_x$:

$$\phi_j^{n+1} = \phi_j^n + \Delta t\left\{-c\delta_{2x}\phi_j^n\right\} + \frac{\Delta t^2}{2}\cdot c^2\delta_x^2\phi_j^n + O(\Delta t^3)$$

i.e.

$$L[\phi_j^n] = \delta_t^F\phi_j^n + c\,\delta_{2x}\phi_j^n - \frac{c^2\Delta t}{2}\delta_x^2\phi_j^n = 0.$$

Truncation error:

$$L[\psi] = \psi_t + \frac{\Delta t}{2}\psi_{tt} + \frac{\Delta t^2}{6}\psi_{ttt} + c\left\{\psi_x + \frac{\Delta x^2}{6}\psi_{xxx}\dots\right\} - \frac{c^2\Delta t}{2}\left\{\psi_{xx}\dots\right\} + O(\Delta x^3, \Delta t^3)$$

$$= \cancel{(\psi_t + c\psi_x)} + \cancel{\left(\frac{\Delta t}{2}\psi_{tt} - \frac{c^2\Delta t}{2}\psi_{xx}\right)} + \frac{\Delta t^2}{6}\psi_{ttt} + \frac{c\Delta x^2}{6}\psi_{xxx}$$

$$= \frac{1}{6}\left(c\Delta x^2 - c^3\Delta t^2\right)\psi_{xxx}\dots \quad \text{(identical to leapfrog).}$$

Again we get exact soln for $\mu = 1$.

Numerical dispersion:

$$\frac{1}{\Delta t}\left[e^{-i\omega\Delta t} - 1\right] = -c\left[\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right] + \frac{c^2\Delta t}{2}\left[\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2}\right]$$

$$e^{-i\omega\Delta t} - 1 = -\mu\left\{i\sin k\Delta x\right\} + \frac{\mu^2}{2}\left\{2\cos k\Delta x - 2\right\}$$

$$\omega = -\frac{1}{i\Delta t}\log\left\{1 - i\mu\sin k\Delta x - \mu^2(1 - \cos k\Delta x)\right\}$$

$$\omega_r \sim ck\left(1 - \frac{(k\Delta x)^2}{6}(1-\mu^2)\dots\right)$$ (No computational mode). (leapfrog phase dispersion)

$\Rightarrow c_{ph}, c_g < c$

$$\omega_i = \frac{1}{2\Delta t}\log\left\{1 - \mu^2(1-\mu^2)(1-\cos k\Delta x)^2\right\}$$ (stable for $\underline{\underline{\mu^2 < 1}}$)

damps short-wavelength waves

$$\sim \frac{1}{2\Delta t}\left\{-\mu^2(1-\mu^2)(k\Delta x)^4/4\right\} \text{ as } k\Delta x \to 0\dots$$

like variable 4th order dissipation.

Bretherton AS 581/Math-AMath 586

Conservation Laws and Finite Volume Methods

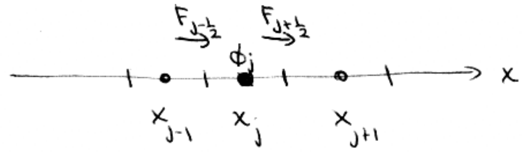
D5.2

Many time-dependent PDEs

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \underbrace{\vec{F}(\psi,\vec{x},t)}_{\text{flux}} = \underbrace{S(\psi,\vec{x},t)}_{\text{source}} \qquad 1D: \frac{\partial \psi}{\partial t} + \frac{\partial F}{\partial x} = S. \quad (*)$$

A natural discretization approach is to divide the domain into cells

$C_j: \; x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$



Integrating $(*)$ w.r.t. x across C_j:

$$\frac{\partial}{\partial t}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \psi \, dx + F(x_{j+\frac{1}{2}},t) - F(x_{j-\frac{1}{2}},t) = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(x,t)\,dx$$

If $\bar{\psi}_j(t) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \psi \, dx$ and $\bar{s}_j = \frac{1}{\Delta x}\int^{x_{j+\frac{1}{2}}} S(x,t)\,dx$, then

$$(*') \quad \frac{\partial \bar{\psi}_j}{\partial t} + \frac{F(x_{j+\frac{1}{2}},t) - F(x_{j-\frac{1}{2}},t)}{\Delta x} = \bar{s}_j \qquad \left(\begin{array}{l}\text{still without}\\ \text{approximation}\end{array}\right)$$

A FDA which follows the form $(*')$, such that $F(x_{j+\frac{1}{2}},t)$ is discretized in terms of approximations ϕ_j^n to values $\bar{\psi}_j(t_n)$, is said to be in "conservation form".

Going one step further, let's consider a 2-level FDA in which $\{\phi_j^{n+1}\}$ are found from $\{\phi_j^n\}$ only, and let's integrate $(*')$ from t_n to $t_{n+1} = t_n + \Delta t$:

$$\bar{\psi}_j(t_{n+1}) - \bar{\psi}_j(t_n) + \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} F(x_{j+\frac{1}{2}},t)\,dt - \int_{t_n}^{t_{n+1}} F(x_{j-\frac{1}{2}},t)\,dt\right] = \int \bar{s}_j \, dt$$

Let ϕ_j^n be a numerical approx to $\bar{\psi}_j(t_n)$

$$F_j^{n+\frac{1}{2}} \quad \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(x_{j+\frac{1}{2}},t)\,dt \qquad \left(\begin{array}{l}\text{the average flux}\\ \text{across } x_{j+\frac{1}{2}} \text{ in } t_n \to t_{n+1}\end{array}\right)$$

(and similarly for $\bar{s}_j$)

Then the FDA will have the form

$$(*'') \quad \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \frac{F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}}{\Delta x} = S_j^{n+\frac{1}{2}} \qquad \left(\begin{array}{c}\text{2-level FDA}\\ \text{in conservation}\\ \text{form}\end{array}\right)$$