- Note that ϕ_j^n is not interpreted as the value $\psi(x_j, t_n)$ but the cell-mean $\overline{\psi}_j(t_n)$, which is different if $\psi(x_j, t_n)$ has $\psi_{xx}''(x_j, t_n) \neq 0$.
- Also note $F_{j+\frac{1}{2}}^n$ is the average flux across $x_{j+\frac{1}{2}}$ in $t_n < t < t_{n+1}$, not the instantaneous flux.

Finite-Volume [Element] or R-E-A methods

D5.5

(1) Reconstruct a form $\tilde{\psi}(x, t_n)$ for the solution from the cell-mean values $\{\phi_j^n\}$

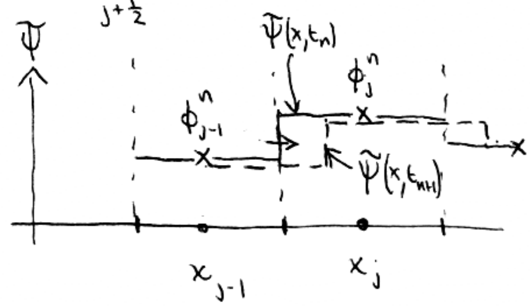
(2) Evolve this solution forward to Δt by exactly or approximately solving the original PDE. This provides the numerical fluxes $F_{j+\frac{1}{2}}^n$

(3) Average the evolved solution $\tilde{\psi}(x, t_{n+1})$ to updated cell-mean values $\{\phi_j^{n+1}\}$

R-E-A with piecewise constant reconstruction for $\psi_t + c\psi_x = 0$.

$(F = c\psi)$

R: $\tilde{\psi}(x, t_n) = \phi_j^n, \quad x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$



E: $\tilde{\psi}(x, t_{n+1}) = \phi_j^n, \quad x_{j-\frac{1}{2}} + c\Delta t < x < x_{j+\frac{1}{2}} + c\Delta t$

$$\left[F_{j-\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} c\,\tilde{\psi}(x_{j-\frac{1}{2}}, t)\,dt\right.$$

$$= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} c\,\phi_{j-1}^n\,dt$$

$$\left. = c\phi_{j-1}^n\right]$$

A: $$\phi_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{\psi}(x, t_{n+1})\,dx = \frac{1}{\Delta x}\left[\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+c\Delta t} \phi_{j-1}^n\,dx + \int_{x_{j-\frac{1}{2}}+c\Delta t}^{x_{j+\frac{1}{2}}} \phi_j^n\,dx\right]$$

$$= \frac{c\Delta t}{\Delta x}\phi_{j-1}^n + \left[1 - \frac{c\Delta t}{\Delta x}\right]\phi_j^n = \phi_j^n - c\Delta t\left\{\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}\right\}$$

$\therefore$ isomorphic to the upwind method.

Equivalently $$\phi_j^{n+1} = \phi_j^n - \Delta t\left\{\frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x}\right\} = \phi_j^n - c\Delta t\left\{\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}\right\}$$ "Flux interpretation"

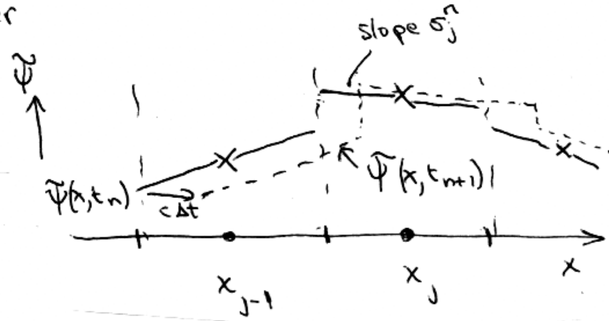
R-E-A with piecewise linear reconstruction

R: $\widetilde{\Psi}(x,t_n) = \phi_j^n + \sigma_j^n (x - x_j)$, $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$

σ_j^n to be specified later

E: $\widetilde{\Psi}(x,t_n+\tau) = \widetilde{\Psi}(x - c\tau, t_n)$,

as before. $\tau = t - t_n$



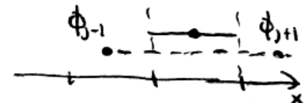
$$\Rightarrow F^n_{j-\frac{1}{2}} = \frac{1}{\Delta t}\int_{t_n}^{t_n+\Delta t} c\,\widetilde{\Psi}(x_{j-\frac{1}{2}},t)\,dt$$

$$= \frac{1}{\Delta t}\int_{t_n}^{t_n+\Delta t} c\cdot\left[\phi^n_{j-1} + \sigma^n_{j-1}(x_{j-\frac{1}{2}} - c\tau - x_j)\right]dt \quad , \tau = t - t_n$$

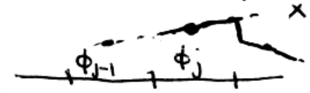
$$F^n_{j-\frac{1}{2}} = c\left[\phi^n_{j-1} + \frac{\sigma^n_{j-1}}{2}\{\Delta x - c\Delta t\}\right]$$

Choice of slopes 3 main possibilities are 2nd order accurate

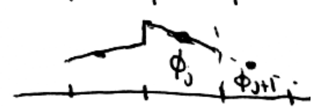
Centered slope: $\sigma_j^n = \dfrac{\phi^n_{j+1} - \phi^n_{j-1}}{2\Delta x}$ (Fromm method)



Upwind slope: $\sigma_j^n = \dfrac{\phi_j^n - \phi^n_{j-1}}{\Delta x}$ (Beam-Warming)



Downwind slope: $\sigma_j^n = \dfrac{\phi^n_{j+1} - \phi^n_j}{\Delta x}$ (Lax-Wendroff)



Can see equivalence with Lax-Wendroff by starting with our earlier L-W formulation:

$$\phi^{n+1}_j = \phi^n_j - \frac{c\Delta t}{2\Delta x}\underbrace{(\phi^n_{j+1} - \phi^n_{j-1})}_{[\phi^n_{j+1}+\phi^n_j]-[\phi^n_j+\phi^n_{j-1}]} + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\underbrace{(\phi^n_{j+1} - 2\phi^n_j + \phi^n_{j-1})}_{[\phi^n_{j+1}-\phi^n_j]-[\phi^n_j-\phi^n_{j-1}]}$$

and writing in flux form

$$\phi^{n+1}_j = \phi^n_j - \frac{\Delta t}{\Delta x}\left[F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right] \quad , F^n_{j-\frac{1}{2}} = c\left[\frac{1}{2}\phi^n_j + \frac{1}{2}\phi^n_{j-1} - \frac{\mu}{2}(\phi^n_j - \phi^n_{j-1})\right]$$

$$= c\left[\phi^n_{j-1} + \frac{1-\mu}{2}(\phi^n_j - \phi^n_{j-1})\right]$$

...which is the same as the slope form above for upwind slope.