Atm S 581 | Amath-Math 586

CLAWPACK

Randy LeVeque has developed a general-purpose software package based on piecewise-linear FV methods for 1D, 2D, and even 3D time-dependent conservation laws. We use it more in Amath 571, but it may be a useful resource for your research or other classes, too. See http://www.amath.washington.edu/~claw .

It was used to generate the plots below and on next page.

diffusive, no osc, no phase errors

much less diffusive, but osc. + phase errors

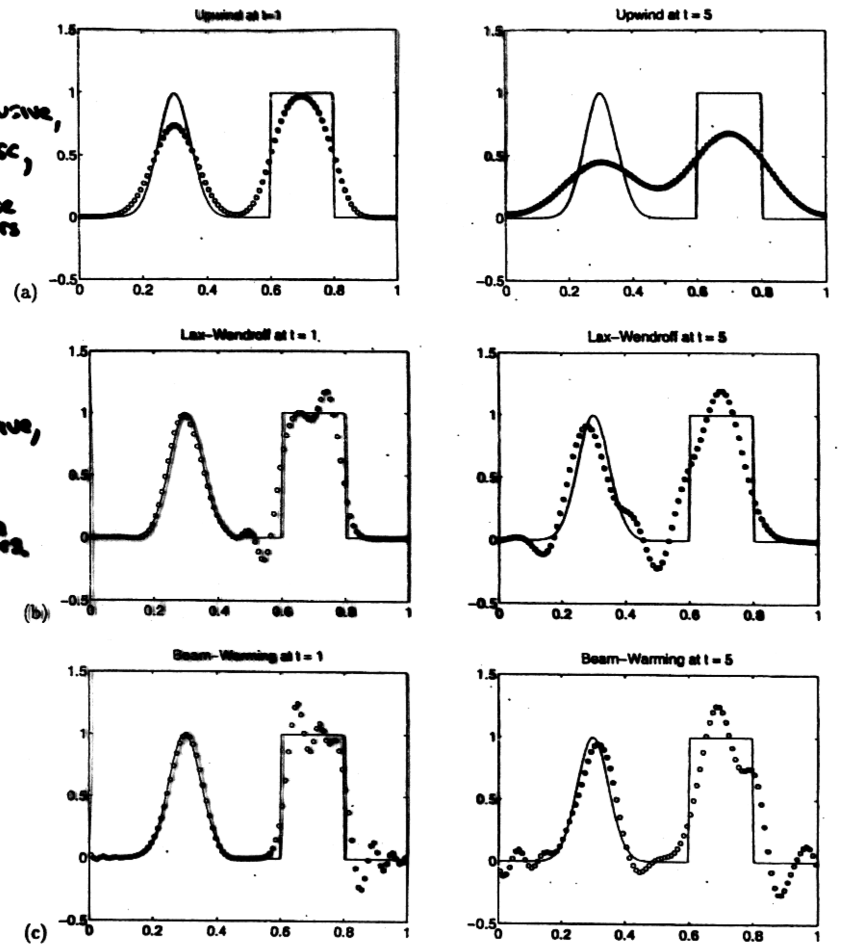


Figure 6.1. Tests on the advection equation with different linear methods. Results at time $t = 1$ and $t = 5$ are shown, corresponding to 1 and 5 revolutions through the domain in which the equation $q_t + q_x = 0$ is solved with periodic boundary conditions. Top: Upwind. Middle: Lax-Wendroff. Bottom: Beam-Warming. [claw/book/chap6/compareadv]

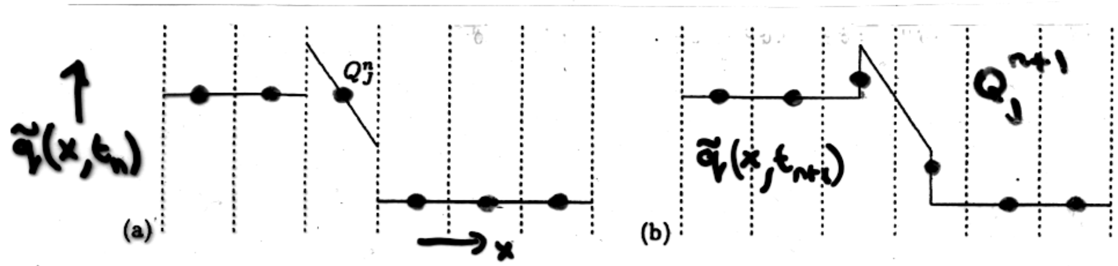
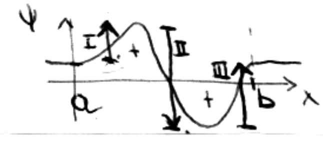


Figure 6.4. (a) Grid values Q^n and reconstructed $\tilde{q}^n(\cdot, t_n)$ using Lax-Wendroff slopes. (b) After advection with $\bar{u}\Delta t = \Delta x/2$. The dots show the new cell averages Q^{n+1}. Note the overshoot.

Slope-limiters to guarantee monotonic/pos. definite advection

A huge advantage of the FV approach is a natural way to suppress spurious oscillations (good for chemical concentrations that physically should remain within given bounds). Mathematically, we may quantify how much a continuous function $\psi(x,t)$ oscillates over a region in terms of its

total variation $\quad TV[\psi] = \int_a^b \left|\frac{\partial \psi}{\partial x}\right| dx .$



Geometrically, TV is the sum of all the upward and downward excursions of $\psi(x,t)$ in $a \le x \le b$, (I) + (II) + (III) in the example above.

Thus if ψ is a solution of the advection eqn. $\psi_t + c\psi_x = 0$,

then $\quad \psi(x,t) = \psi(x-ct,0)$ [waveforms will just be translated a distance ct] so on a periodic domain,

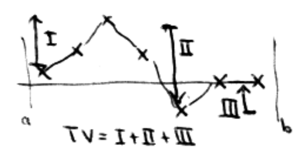
$$TV[\psi(x,t)] = \int_a^b \psi(x,t)\,dx = \int_a^b \psi(x-ct,0)\,dx = \int_a^b \psi(x,0)\,dx$$

[by periodic rearrangement of ψ].

is conserved. Similarly on an infinite domain.

For a discrete method, the analogous quantity is

$$TV[\vec{\phi}^{\,n}] = \sum_{j=1}^{N} |\phi_j^n - \phi_{j-1}^n|$$



TV = I + II + III

A discrete method can be assured of not introducing spurious oscillations that increase TV if it is total-variation diminishing (TVD),

$$TV[\vec{\phi}^{\,n+1}] \le TV[\vec{\phi}^{\,n}] \qquad \text{for advection eqn.}$$

In particular, TVD methods are monotonicity-preserving or "monotonic", i.e. for monotone initial data $\phi_j^n \ge \phi_{j+1}^n \ \forall j$, $TV[\vec{\phi}^{\,n}] = \phi_N - \phi_0$

so if we consider a ramp with $\phi_j = \phi_0$, $j \le 0$ and $\phi_j = \phi_N$, $j \ge N$, and a TVD method

$$TV[\vec{\phi}^{\,n+1}] \le TV[\vec{\phi}^{\,n}]$$

but if $\vec{\phi}_j^{\,n+1} \to \phi_0$ as $j \to -\infty$

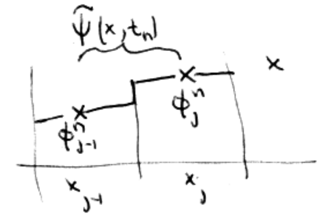
$\phi_j^{n+1} \to \phi_N$ as $j \to +\infty$

then $\quad TV[\vec{\phi}^{\,n+1}] = \phi_N - \phi_0 +$ TV of new internal oscillations

so TVD $\Rightarrow$ TV(new internal oscillations) $= 0$.

So, can we construct a FV-REA method that is TVD?

Yes! If the reconstructed function $\tilde{\Psi}(x,t_n)$ is monotonic between each pair of gridpoints (x_{j-1}, x_j), this is a sufficient condition for TVD.



Proof: (1) For such a reconstruction

$$\underset{x_{j-1} \le x \le x_j}{TV}\left[\tilde{\Psi}(x,t_n)\right] = \left|\phi_j^n - \phi_{j-1}^n\right|$$

so

$$\underset{\substack{-\infty < x < \infty \\ \text{or periodic}}}{TV}\left[\tilde{\Psi}(x,t_n)\right] = \sum_j \left|\phi_j^n - \phi_{j-1}^n\right| = TV\left[\vec{\phi}^n\right]$$

(2) Evolving to $\tilde{\Psi}(x,t_{n+1})$ does not change TV since it is just advection of reconstructed profile to right by $c\Delta t$

(3) Averaging $\tilde{\Psi}(x,t_{n+1})$ across gridcells C_j can only reduce TV

$$\Rightarrow \quad TV\left[\vec{\phi}^{n+1}\right] \le TV\left[\vec{\phi}^n\right].$$

A variety of reconstruction schemes with linear slopes satisfy this constraint, so are TVD, but are also 2nd-order accurate for monotone data. Example:

minmod slope limiter: $$\sigma_j^n = \text{minmod}\left(\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}, \frac{\phi_{j+1}^n - \phi_j^n}{\Delta x}\right)$$

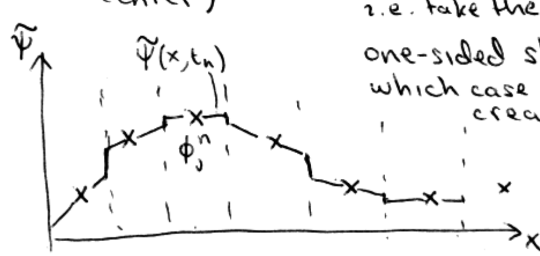
$$\text{minmod}(a,b) = \begin{cases} a & , |a| < |b| \text{ and } a,b \text{ have same sign} \\ b & , |b| < |a| \quad " \quad " \quad " \quad " \\ 0 & , a,b \text{ have opposite signs.} \end{cases}$$

i.e. take smaller of two one-sided slopes

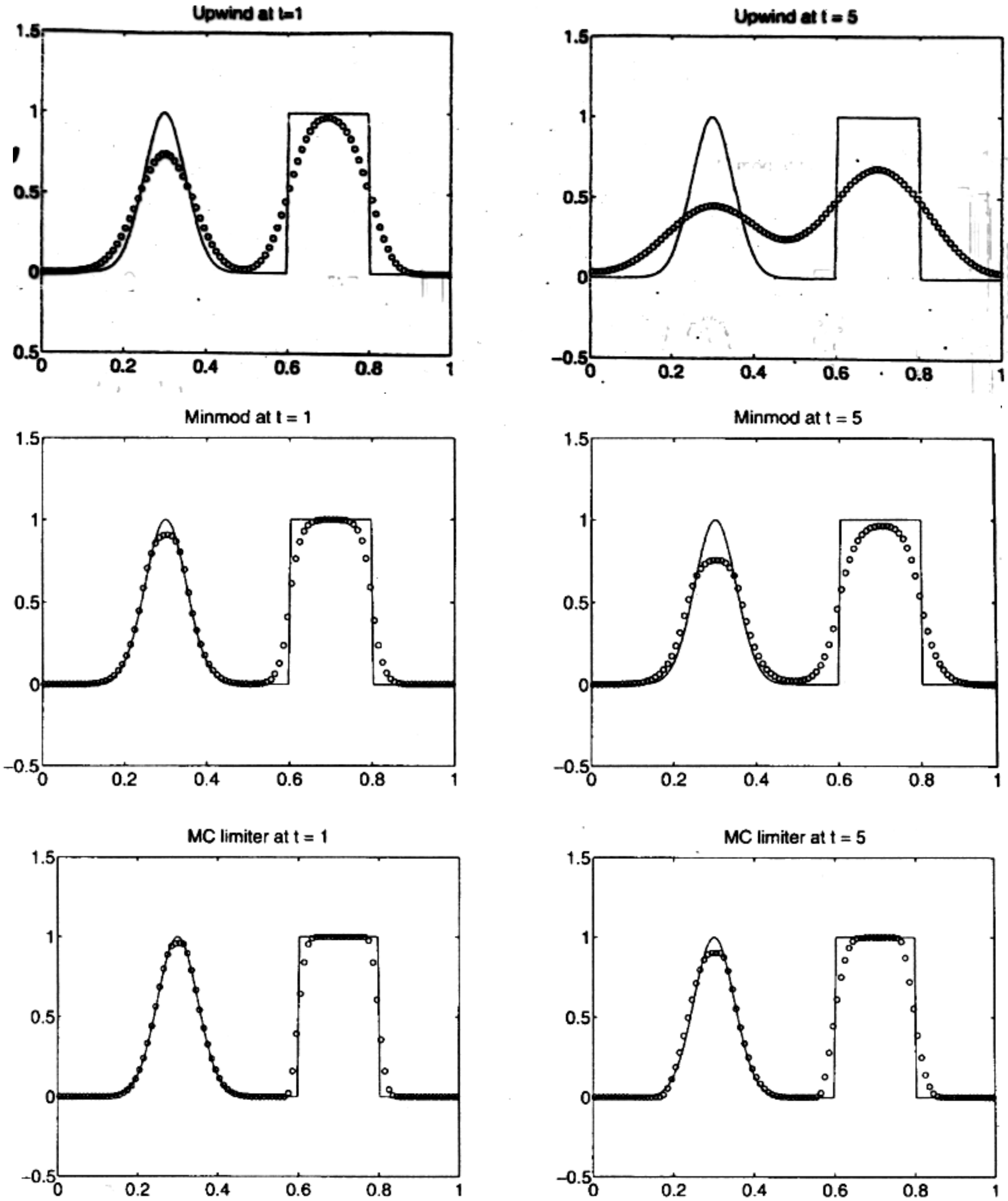
More accurate:

MC-limiter (Monotonized center)

$$\sigma_j^n = \text{minmod}\left(\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x}, 2\,\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}, 2\,\frac{\phi_{j+1}^n - \phi_j^n}{\Delta x}\right)$$



i.e. take the "centered" slope unless one of the one-sided slopes is less than half as large, in which case the centered slope would possibly create a new extremum. At an extremal ϕ_j^n, slope $\sigma_j^n = 0$.



Solutions to advection equation after 1 (left) and 5 (right) time units using different TVD REA methods. The MC slope limiter provides best overall performance.

Accuracy of slope-limiter methods

On smooth fields such as a sine wave, Lax-Wendroff, Beam-Warming and Fromm are 2^{nd}-order accurate. Slope-limiter methods such as MC are more accurate for coarsely-resolved oscillations but have slightly lower-order convergence (~1.7-order for a sinusoid). The top panels of the plot below show this behavior; the plot was generated by script LW_MC_errconv.m on the class web page. The left panel shows the reconstructed functional form of the approximate solution using both methods for the coarsest resolution tested. It shows how LW gives a nonmonotonic reconstruction between gridpoints, while MC gives a monotonic reconstruction between gridpoints that guarantees it is TVD.

On fields with sharp discontinuities, MC gives lower rms errors compared to Lax-Wendroff in addition to preventing overshoots. Both methods converge much more slowly, because regardless of resolution the rms errors are dominated by poorly-resolved grid-scale wavenumbers.