Applying FD and FV methods to more complex equations

For fluid simulation, a typical PDE we might want to solve is:

$$\frac{\partial \psi}{\partial t} + \vec{u}(x,y,t)\cdot\nabla\psi = a\nabla^2\psi + S_\psi$$

where unknown $\psi(x,y,t)$, $\vec{u}$ is a known velocity field satisfying $\nabla\cdot\vec{u} = 0$, and S_ψ is some source term.

Involves 3 complications over 1D advection eqn:

(1) Flow velocity $\vec{u}$ depends on x, t
(2) Source terms
(3) Multiple space dimensions

In a standard finite difference method, these are easily handled

- Each term is discretized in space (say using centered or upwind differencing) to a consistent order of accuracy.
- A time differencing method is chosen for stability, efficiency, and consistent order of accuracy to space differencing. If S_ψ is nonlinear in ψ, or $\vec{u}$ depends on ψ, or we have multiple space dimensions, explicit time differencing will be easier to implement than an implicit method such as trapezoidal.

Example Burgers eqn

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = a\frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l} a = 0.01 \\ u(x,t) \text{ periodic on } 0<x<1 \\ u(x,0) = 1 + \sin(2\pi x). \end{array}$$

burgers.m on web page:
- FD using centered space differencing on $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$
- RK4 for time difference
- Uniform grid $x_j = j\Delta x$, $\Delta x = \frac{1}{N}$

Works fine if Δx fine enough to resolve developing shock, but note timestep limitations for RK4 stability

$$|\sigma_{i,max}\Delta t| = \left|\frac{u_{max}\Delta t}{\Delta x}\right| < 2.82$$

$$|\sigma_{r,max}\Delta t| = \left|\frac{4a\,\Delta t}{\Delta x^2}\right| < 2.82$$

For $\Delta x < \frac{4a}{U_{max}} = \frac{4(0.01)}{2} = 0.02$, the diffusive term (not the advective term) limits the ~~stability~~ timestep Δt.

We know the Crank-Nicolson method could avoid this stability limit on Δt, but the uu_x term makes trapezoidal time differencing impractical. Is there anything we can do to use C-N on the diffusive term, RK4 on the advection?

<u>Splitting</u> (Durran 4.3)

Consider the PDE $\frac{\partial \psi}{\partial t} = A(\psi) + B(\psi)$

For simplicity, assume A and B are linear, time-independent operators (not true in above example!)

Then
$$\psi^{n+1} = \exp\{\Delta t(A+B)\}\psi^n$$
$$= 1 + \Delta t(A+B)\psi^n + \frac{\Delta t^2}{2}\underbrace{(A+B)(A+B)}_{A^2+AB+BA+B^2}\psi^n \ldots$$

Now try <u>splitting</u> into two sequential updates:

$$\frac{\partial \psi^*}{\partial t} = A(\psi) \text{ then } \frac{\partial \psi}{\partial t} = B(\psi):$$
$$\Rightarrow \psi^{*} = \exp(\Delta t A)\psi = (1 + \Delta t A + \frac{\Delta t^2}{2}A^2 \ldots)\psi$$
$$\hat{\psi}^{n+1} = \exp(\Delta t B)\psi^* = (1 + \Delta t B + \frac{\Delta t^2}{2}B^2 \ldots)\psi^*$$

This would give
$$\hat{\psi}^{n+1} = \exp(\Delta t B)\exp(\Delta t B)\psi^n$$
$$= (1 + \Delta t B + \frac{\Delta t^2}{2}B^2 \ldots)(1 + \Delta t A + \frac{\Delta t^2}{2}A^2 \ldots)\psi^n$$
$$= (1 + \Delta t(A+B) + \frac{\Delta t^2}{2}(A^2 + 2BA + B^2))\psi^n \ldots$$
$$\neq \text{exact } \psi^{n+1} \text{ <u>unless</u> } AB\psi = BA\psi \text{, i.e. } A, B \text{ <u>commute</u>}$$

The "splitting error" appears to be $O(\Delta t^2)$ per timestep, corresponding to 1st-order accuracy, to the extent A, B don't commute. If A, B commute, e.g. $A = \delta_x^2$, $B = \delta_y^2$, then no splitting error.

Strang splitting

The formal splitting error can be reduced by

(1) Applying A for $\Delta t/2$

(2) Applying B for Δt "strang splitting"

(3) Applying A for $\Delta t/2$

which gives

$$\hat{\psi}^{n+1} = \exp\left(\frac{\Delta t}{2} A\right) \exp(\Delta t B) \exp\left(\frac{\Delta t}{2} A\right) \psi^n$$

$$= \left(1 + \frac{\Delta t}{2} A + \frac{\Delta t^2}{8} A^2 \ldots\right)\left(1 + \Delta t B + \frac{\Delta t^2}{2} B^2\right)\left(1 + \frac{\Delta t}{2} A + \frac{\Delta t^2}{8} A^2 \ldots\right)\psi^n$$

$$= 1 + \Delta t(A+B) + \frac{\Delta t^2}{2}\left(\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right]A^2 + AB + BA + \left[\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right]B^2\right) + \ldots \quad \psi^n$$

$$= \psi^{n+1} + O(\Delta t^3)$$

$\Rightarrow$ Strang splitting is 2^{nd}-order accurate in Δt even if $AB \neq BA$.

Equivalence of sequential + Strang splitting

Sequential:

$$\hat{\psi}^n = \underbrace{e^{\Delta t B} e^{\Delta t A} \; e^{\Delta t B} e^{\Delta t A} \ldots e^{\Delta t B} e^{\Delta t A}}_{n \text{ times}} \psi^0$$

Strang:

$$\hat{\psi}^n = \underbrace{e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A} \; e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A} \ldots e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A}}_{n \text{ time's}} \psi^0$$

$$= e^{\frac{\Delta t}{2} A} \underbrace{e^{\Delta t B} e^{\Delta t A} \; e^{\Delta t B} e^{\Delta t A} \ldots}_{n-1 \text{ times}} e^{\Delta t B} e^{\frac{\Delta t}{2} A} \psi^0$$

Thus Strang is like doing $\frac{1}{2}$ step of A, then $n-1$ steps of sequential splitting, then a step of B but only $\frac{1}{2}$ step of A. This differs from n steps of sequential only by $O(\Delta t^2)$ so in fact <u>sequential splitting is also $O(\Delta t^2)$ accurate</u>!

See burgers_split.m for sequential splitting applied to Burgers eqn with RK4 for advection, C-N for the diffusion, and a Δt twice as large as for unsplit method for $N_x = 100$ pts.