Now, let $x_j = (j-1)\Delta x$, $\Delta x = \frac{L}{N}$ be N evenly spaced gridpoints over $[0, L]$ (excluding the periodic point $x_{N+1} = L$). Then

$$
\begin{aligned}
Q_j(t) \equiv Q(x_j, t) &= \sum_{n=1}^{N} \hat{q}_n(t)\, \varphi_n(x_j) \\
&= \frac{1}{N}\sum_{n=1}^{N} \hat{q}_n(t) \cdot e^{ik_n x_j} \\
&= \frac{1}{N}\left\{ \sum_{n=1}^{N/2} \hat{q}_n(t) \exp\left\{ i \cdot \frac{2\pi}{L}(n-1) \cdot \frac{L}{N}(j-1) \right\} \right. \\
&\qquad \left. + \sum_{n=\frac{N}{2}+1}^{N} \hat{q}_n(t) \exp\left\{ i \cdot \frac{2\pi}{L}(n-1-N) \cdot \frac{L}{N}(j-1) \right\} \right\} \\
&= \frac{1}{N}\left\{ \sum_{n=1}^{N/2} \hat{q}_n(t) \exp\left\{ \frac{2\pi i}{N}(j-1)(n-1) \right\} \right. \\
&\qquad \left. + \sum_{n=\frac{N}{2}+1}^{N} \hat{q}_n(t) \exp\left\{ \frac{2\pi i}{N}(j-1)(n-1) \right\} \cancelto{1}{\exp\left\{ -\frac{2\pi i N}{N}(n-1) \right\}} \right\} \\
&= \frac{1}{N}\sum_{n=1}^{N} \hat{q}_n(t) \exp\left\{ \frac{2\pi i}{N}(j-1)(n-1) \right\}
\end{aligned}
$$

i.e. $\quad \{Q_j\} = \text{IDFT}\{\hat{q}_n\} \iff \{\hat{q}_n(t)\} = \text{DFT}\{Q_j(t)\}$.

The discrete Fourier coefficients can thus be derived as a DFT of the gridpoint values of Q at the N gridpoints x_j. In practice, we try to choose N to be a power of 2, so we can use a efficient FFT (fast Fourier transform) algorithm for the DFT and IDFT.

Returning to the residual $R[Q] = Q_t - S(Q)$, if we express $R = \sum_{n=1}^{N} \hat{r}_n(t)\varphi_n(x)$ and let $R_j = R[Q(x_j)]$, then the Galerkin method would be to require that $\hat{r}_n(t) = 0$ and the collocation method would require $R_j = 0$, $j = 1, \dots, N$.

Since $\{\hat{r}_n\} = \text{DFT}\{R_j\}$ these are actually equivalent methods, if $S(Q)$ is a linear combo of the φ_n's, $n = 1, \dots, N$. (but not if $S(Q)$ is a nonlinear or non-constant coeff in x operator).

Example of differentiation with DFT

$\mathbf{y}$ = N-vector defined at gridpoints $x_j = (j\text{-}1)\Delta x$,
$j = 1,\ldots,N$, $\Delta x = L/N$.

Matlab code for derivative

```
Y = fft(y);
M = [0:(N/2-1) (-N/2):-1];
k = 2*pi*M/L;
dydx = real(ifft(1i*k.*Y))
```

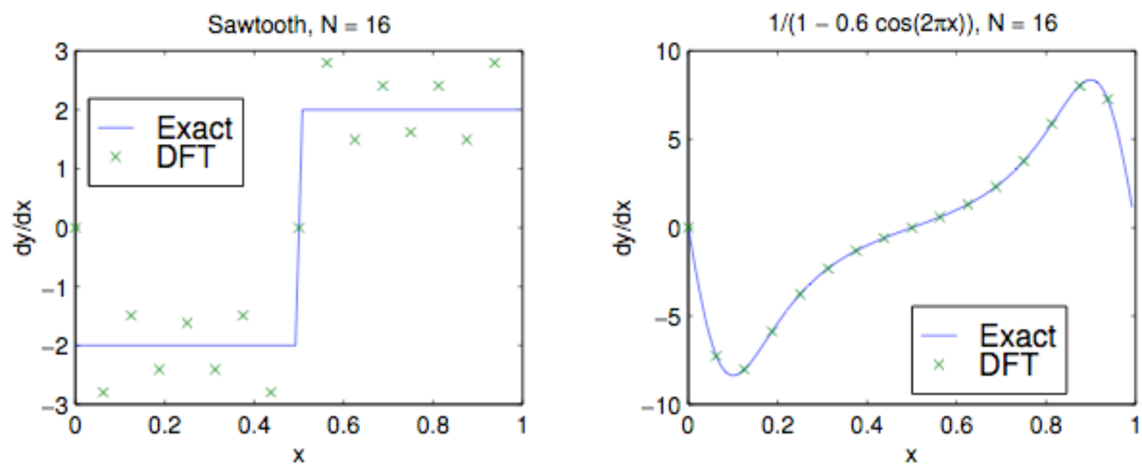

Figure 2: Exact and DFT-computed derivative of [0, 1] periodic sawtooth and swell functions.

Relation to coefficients of conventional complex Fourier series

The DFT $\{\hat{y}_n, n = 1,\ldots,N\}$ of an L-periodic function $y(x)$ gives approximations to the coefficients $\tilde{y}_M$ in its complex Fourier series

$$y(x) = \sum_{M=-\infty}^{\infty} \tilde{y}_M \exp(2\pi i M x/L), \quad \tilde{y}_M = L^{-1}\int_0^L y(x)\exp(-2\pi i M x/L)dx$$

If we approximate the integral as a Riemann sum over intervals of width Δx centered on x_j, $j = 1,\ldots,N$,

$$\tilde{y}_M \approx L^{-1}\sum_{j=1}^{N} y_j \exp(-2\pi i M x_j/L)\Delta x = \frac{1}{N}\sum_{j=1}^{N} y_j \exp(-iK_n x_j) = \frac{\hat{y}_n}{N}$$

where $M = \begin{cases} n-1 & n \le N/2 \\ n-1-N & N/2+1 \le n \le N \end{cases}$ is defined as for the derivative.

Spectral accuracy

Recall that for an L-periodic, $C^\infty[0,L]$ function $y(x)$, the representation (F) in terms of N complex Fourier modes approximates $y(x)$ and all its derivatives with accuracy $o(N^{-r})$ for any r, i.e. with accuracy exceeding any power of N. Usually the error (L^2 norm) is $O(e^{-\alpha N})$ for some α. This makes spectral methods extraordinarily efficient in representing smooth functions and their derivatives accurately with few spectral modes. If $y(x)$ is nonsmooth, the spectral representation is less efficient. For instance, if $y(x)$ has step-like jumps and is otherwise continuous (i.e. piecewise continuous), the L^2-norm error in representing $y(x)$ with N Fourier modes is $O(N^{-1})$. This is one reason FD|FV|FE methods are popular for problems involving sharp gradients, even in simple geometries.