Beginning of lecture was taken from pp. 2-4 of DFT notes, which discuss how the coefficients of the DFT are proportional to Riemann-sum approximations to the Fourier coefficients for the complex Fourier coefficients indexed $-M/2 \le N < M/2$, and discussed that if $y(x)$ is L-periodic and has a p'th derivative that is piecewise continuous with bounded total variation, then the Fourier and DFT coeffs are O($K_M^{-(p+1)}$), where $K_M = 2\pi M/L$. More detail: Chapter 4 of Trefethen, 2000, *Spectral Methods in Matlab*, SIAM Press.

Efficiency of Fourier representation $y(x) = \sum_{M=-\infty}^{\infty} \hat{y}_M \exp(2\pi i M x / L)$ on periodic domain $0 < x < L = 1$:

Examples

Step ($p = 0$):

$$y(x) = \mathrm{sgn}(x - 0.5)$$

$$\hat{y}_M = \frac{2i}{\pi M}, \quad M \text{ odd}$$

Saw ($p = 1$):

$$y(x) = |2x - 1| - 0.5$$

$$\hat{y}_M = \frac{2}{\pi^2 M^2}, \quad M \text{ odd}$$

Swell ($p = \infty$):

$$y(x) = [1 - 0.6\cos(2\pi x)]^{-1}$$

$$\hat{y}_M = \frac{5}{4} 3^{-|M|}$$

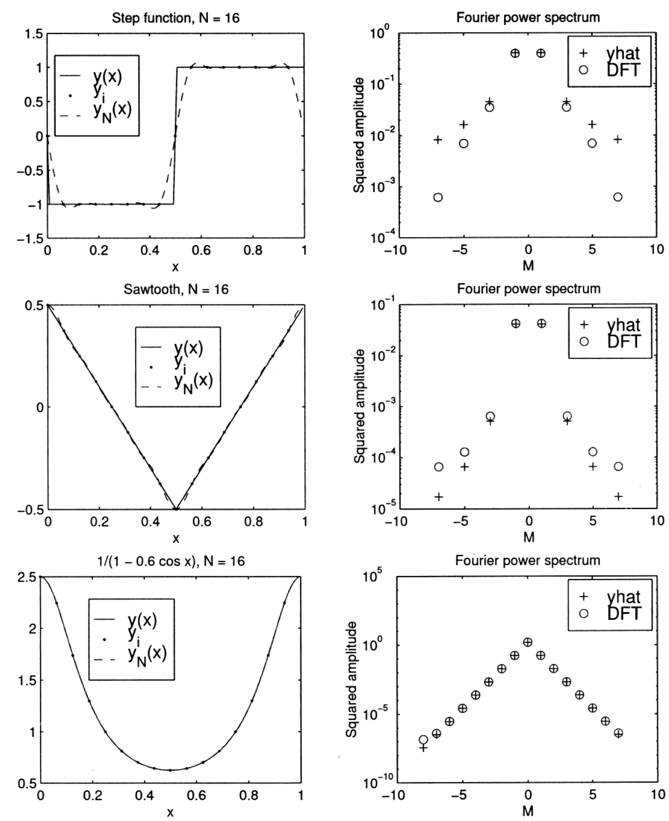

Figure 1: Left panels: Function $y(x)$ (solid), grid points y_i, and FFT-based interpolating function $y_N(x)$. Right panels: Power spectrum of Fourier components (+) and of DFT (normalized by dividing DFT amplitudes by N).

To 'prove' the Fourier coeffs behave this way, we integrate the formula for $\hat{y}_M$ by parts:

$$\hat{y}_M = \frac{1}{L}\int_0^L y(x)\exp(-iK_M x)\,dx$$

$$= \left.\frac{y\exp(-iK_M x)}{-iK_M}\right|_0^L + \frac{1}{iK_M}\frac{1}{L}\int_0^L y'(x)\exp(-iK_M x)\,dx$$

$$= \left.\frac{y'\exp(-iK_M x)}{(-iK_M)^2}\right|_0^L + \left(\frac{1}{iK_M}\right)^2 \frac{1}{L}\int_0^L y''(x)\exp(-iK_M x)\,dx \quad \ldots$$

…

At each step, the boundary terms cancel to zero due to the periodic BC. This can be repeated as often as $y(x)$ is differentiable. By carrying it out p times and estimating the integral we obtain

$$\hat{y}_M = \left(\frac{1}{iK_M}\right)^p \frac{1}{L}\int_0^L y^{(p)}(x)\exp(-iK_M x)\,dx$$

If $y(x)$ is infinitely differentiable: we can carry this integration by parts out indefinitely. For any p, the absolute value of the integrand is bounded and independent of KM, so the Fourier coefficients go to zero at least as fast as K_M^{-p}. Thus the Fourier coefficient must go to zero faster than any power of K_M

If $y^{(p)}(x)$ is piecewise continuously with bounded total variation (like the square wave with p = 0 or the sawtooth with p = 1), we can divide $[0, L]$ into M subintervals (some of which may have zero length), each containing an equal fraction of the total variation. In each segment, we separate $y^{(p)}(x)$ in the above integral into its mean value across the segment plus a small residual. Summed over all the segments, and noting K_M is proportional to M, each of these terms can be estimated to give a contribution $O(K_M^{-1})$ to the integral so the Fourier coefficient is $O(K_M^{-(p+1)})$.

Fourier Spectral Method on 1D advection equation

Consider:

$$q_t + cq_x = 0 \text{ on } 0 < x < L$$

Periodic BCs

IC: $q(x,0) = q_0(x)$

We find an approximate solution $Q(x, t)$ using N Fourier modes:

$$Q(x,t) = N^{-1}\sum_{n=1}^{N}\hat{q}_n(t)\exp(ik_n x)\,, \quad k_n = \frac{2\pi}{L}[0,\ldots,\frac{N}{2}-1,-\frac{N}{2},\ldots,-1] \text{ for } n = 1,\ldots,N.$$

Using N equally spaced gridpoints $x_j = (j-1)L/N$ we set

$$\{\hat{q}_n(0)\} = \text{DFT}\{q_0(x_j)\}$$

Using collocation, we insist on zero residual at the gridpoints:

$$0 = R_j(t) = \left[Q_t + cQ_x\right](x_j, t)$$

Taking the DFT of the residual vector $\{R_j\}$, and evaluating the derivative on the RHS using the DFT:

$$0 = \frac{d\hat{q}_n}{dt} + ick_n\hat{q}_n \quad \Rightarrow \frac{d\hat{q}_n}{dt} = -i\omega_n\hat{q}_n, \quad \omega_n = k_n c \quad \text{for } n = 1,\ldots,N$$

We could solve this equation analytically, which would yield a solution whose accuracy is limited only by how well the DFT can represent $q_0(x)$:

$$\hat{q}_n(t) = \hat{q}_n(0)\exp(-ik_n ct)$$

$$Q(x,t) = N^{-1}\sum_{n=1}^{N}\hat{q}_n(t)\exp(ik_n x) = N^{-1}\sum_{n=1}^{N}\hat{q}_n(0)\exp(ik_n\left[x-ct\right]) \approx q_0(x-ct)$$

However, in a real problem this would not be possible (else we'd be solving the original PDE analytically), so in the next lecture we discuss good time-differencing methods for FS methods.