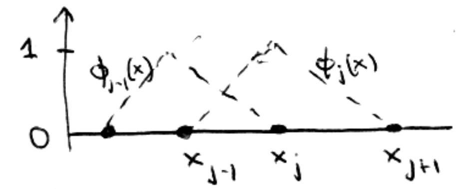
Finite Element Method (Durran 4.5)

Use localized, nonorthogonal basis fns, e.g. the "chapeau" or hat functions

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{\Delta x_{j-\frac{1}{2}}}, & x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{\Delta x_{j+\frac{1}{2}}}, & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$



$\Delta x_{j-\frac{1}{2}} = x_j - x_{j-1}$

--> FEM advantage: No need to use a uniform grid

Suppose we wish to solve the time-dependent PDE

$$\psi_t + S^{(x)}[\psi] = 0 \qquad (\text{e.g. } S^{(x)}[\psi] = c\psi_x \text{ for advection})$$

on a grid $\{x_j\}$. Let

$$\phi(x,t) = \sum_j a_j(t)\,\phi_j(x)$$

be a Galerkin approx to $\psi(x,t)$. Then to calculate a_j's we find residual

$$R = \phi_t - S^{(x)}[\phi] = \sum_n \frac{da_n}{dt}\phi_j - \sum_j S^{(x)}\Big[\sum a_n(t)\phi_n\Big]$$

and take inner prod with each ϕ_j; where $\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx$:

$$0 = \langle \phi_j, R\rangle = \sum_{n=1}^{N} \dot{a}_n I_{nj} + \langle \phi_j, S^{(x)}[\phi]\rangle, \quad \dot{a}_n = \frac{da_n}{dt}$$

Now $I_{nj} = \langle \phi_j, \phi_n\rangle = 0$ if $n \ne j-1, j, j+1$

$$I_{j-1,j} = \langle \phi_{j-1}, \phi_j\rangle = \int_{0 \leftrightarrow x_{j-1}}^{1 \leftrightarrow x_j} \underbrace{(1-\xi)}_{\phi_{j-1}}\,\underbrace{\xi}_{\phi_j}\,\underbrace{\Delta x_{j-\frac{1}{2}}\,d\xi}_{dx} \quad \text{where } \xi = \frac{x - x_{j-1}}{\Delta x_{j-\frac{1}{2}}}$$

$$= \frac{1}{6}\Delta x_{j-\frac{1}{2}}$$

$$I_{j,j} = \langle \phi_j, \phi_j\rangle = \int_0^1 \xi\cdot\xi\,\Delta x_{j-\frac{1}{2}}\,d\xi - \int_1^0 \zeta\cdot\zeta\,\Delta x_{j+\frac{1}{2}}\,d\zeta, \quad \zeta = \frac{x_{j+1} - x}{\Delta x_{j+\frac{1}{2}}}$$

$$= \frac{1}{3}\left(\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}}\right)$$

$$I_{j,j+1} = \langle \phi_{j+1}, \phi_j\rangle = \langle \phi_j, \phi_{j+1}\rangle$$

$$= \frac{1}{6}\Delta x_{j+\frac{1}{2}}$$

Once $\langle \phi_j, S^{(x)}\rangle$ is written in terms of the a_n's, this is a tridiagonal system for the $\dot{a}_n$'s in terms of the a_n's.

Example: Constant advection $S^{(x)}[\psi] = c\psi_x$

$$\Rightarrow \langle \phi_j , S^{(x)}[\phi] \rangle = \sum_n a_n \underbrace{\langle \phi_j , c\frac{d\phi_n}{dx} \rangle}_{J_{jn}}$$

Here

$$J_{jn} = 0 \quad \text{if } n \neq j-1, j, j+1$$

$$J_{j,j-1} = \langle \phi_j , c\frac{d\phi_{j-1}}{dx} \rangle = c\int_0^1 \underbrace{\xi}_{\phi_j} \underbrace{\left(-\frac{1}{\Delta x_{j-\frac{1}{2}}}\right)}_{d\phi_{j-1}/dx} \underbrace{\Delta x_{j-\frac{1}{2}} d\xi}_{dx}$$
$$= -c/2$$

$$J_{j,j} = \langle \phi_j , c\frac{d\phi_j}{dx} \rangle = c\int_0^1 \xi\left(\frac{1}{\Delta x_{j-\frac{1}{2}}}\right)\Delta x_{j-\frac{1}{2}} d\xi + c\int_1^0 \eta\left(-\frac{1}{\Delta x_{j+\frac{1}{2}}}\right)(-\Delta x_{j+\frac{1}{2}} d\eta)$$
$$= 0$$

$$J_{j,j+1} = \ldots \text{similarly}, \ \frac{c}{2}$$

so

$$\langle \phi_j , S^{(x)}[\phi] \rangle = \frac{c}{2}\left[a_{j+1} - a_{j-1}\right]$$

and our FEM is the implicit system

$$\frac{1}{6}\Delta x_{j-\frac{1}{2}}\dot{a}_{j-1} + \frac{1}{3}\left(\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}}\right)\dot{a}_j + \frac{1}{6}\Delta x_{j+\frac{1}{2}}\dot{a}_{j+1} + \frac{c}{2}\left[a_{j+1} - a_{j-1}\right] = 0$$

for each j.

Initial/boundary conditions

Since the $a_j(0)$ are just the gridpoint values of ϕ at $t=0$, an IC $\psi(x,0) = f(x)$ implies

$$a_j(0) = f(x_j)$$

The advection eqn requires an upstream BC if domain is not periodic. If we are solving $\psi_t + c\psi_x = 0$ on $0 < x < L$ with $\psi(0,t) = g(t)$, we use a boundary node with basis function

$$\phi_0(x) = \begin{cases} 1 - \frac{x}{\Delta x_{\frac{1}{2}}}, & 0 = x_0 \le x \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

such that

$$\phi(x,t) = \underbrace{a_0(t)}_{= g(t) \text{ by BC}}\phi_0(x) + \sum_{n=1}^{N} a_n(t)\phi_n(x)$$

Then $\langle \phi_j , \underbrace{c\phi_x}_{S^{(x)}[\phi]} \rangle = a_0 J_{j0} + \sum_{n=1}^{n} a_n J_{jn}$

$$J_{j0} = \left\langle \phi_j , c\frac{d\phi_0}{dx} \right\rangle = \begin{cases} 0 & \text{if } j > 1 \\ \frac{c}{2} & \text{if } j = 1 \end{cases}$$

and for $j=1$

$$\frac{1}{6}\Delta x_{\frac{1}{2}} \dot{a}_0 + \frac{1}{3}\left(\Delta x_{\frac{1}{2}} + \Delta x_{\frac{3}{2}}\right)\dot{a}_1 + \frac{1}{6}\Delta x_{\frac{3}{2}} \dot{a}_2 + \frac{c}{2}\left[a_2 - \overset{g(t)}{a_0}\right] = 0$$

Thus, with N nodes we obtain the system

$$I\dot{\vec{a}} + J\vec{a} = \vec{r} = \begin{bmatrix} c/2 \cdot g(t) - \frac{\Delta x_{1/2}}{6}\dot{g}(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This can be solved using any time-differencing method; Matlab ODE solvers allow this type of implicit ODE system.

Trapezoidal time differencing:

$$I\left\{\frac{a^{n+1} - a^n}{\Delta t}\right\} + J\left\{\frac{a^{n+1} + a^n}{2}\right\} = \vec{r}^{\,n+\frac{1}{2}}$$

$$\left(\frac{I}{\Delta t} + \frac{J}{2}\right)\vec{a}^{\,n+1} = \left(\frac{I}{\Delta t} - \frac{J}{2}\right)\vec{a}^{\,n} + \vec{r}^{\,n+\frac{1}{2}}$$

... a tridiagonal linear system for $\vec{a}^{\,n+1}$.

See Matlab function advect_FEMsolve.m for an implementation of this algorithm, and advect_FEM.m for an application to the IC $\phi(x,0) = 0$ and BC $\phi(0,t) = g(t) = \sin(50t)$, with $\Delta x = 0.01$, $\Delta t = 0.005$, and a plot of the numerical and exact solns at $t = 0.9$.