Aliasing

Consider a wavenumber k with $|k| > \pi/\Delta x$. Look at $\mathrm{Re}[e^{ikx}] = \cos kx$, e.g. for $k = \frac{3}{2}\frac{\pi}{\Delta x}$, $\lambda = \frac{2\pi}{k} = \frac{4}{3}\Delta x$. Sampled at the gridpoints k_j it appears identical to a lower wavenumber. In general, if

$$k = k_a + \frac{2\pi}{\Delta x}\cdot m,$$

$$e^{ikx_j} = e^{ik_a x_j}\, e^{\frac{2\pi}{\Delta x}\cdot m \cdot j\Delta x} = e^{ik_a x_j}$$

, so any waveno. is identical ("aliased") to one shifted by a multiple of $\frac{2\pi}{\Delta x}$ into $-\frac{\cancel{2}\pi}{\Delta x} \le k < \frac{\cancel{2}\pi}{\Delta x}$.

Parseval Theorem for DFT

$$\|\phi\|_2^2 = \Delta x\, |\underset{\sim}{\phi}|^2 = \Delta x(\phi^\dagger \phi)$$

$$= \Delta x\,(\underset{\sim}{a}^\dagger F^\dagger \cdot F\underset{\sim}{a})$$

$$= \Delta x\,(\underset{\sim}{a}^\dagger \underset{\sim}{a}) = \|a\|_2^2 .$$

Sufficiency of Von Neumann Condition

Suppose that $|A(k_m)| \le 1 + \gamma\Delta t$ for all k_m resolved on grid.

Then
$$\|\phi^n\|_2^2 = \Delta x \sum_m |a_m^n|^2 = \Delta x \sum_m |A(k_m)|^{2n}\, |a_m^0|^2$$

$$\le (1+\gamma\Delta t)^{2n}\, \Delta x \sum_m |a_m^0|^2$$

$$= (1+\gamma\Delta t)^{2n}\, \|\phi^0\|_2^2$$

$$\le e^{\gamma\cdot 2n\Delta t}\, \|\phi^0\|_2^2$$

which satisfies our defn of exponential stability, with $\beta = \gamma$

CFL condition for hyperbolic PDE's (D2.2.3)

The numerical domain of dependence of a consistent linear FDA must include the domain of dependence of the associated hyperbolic PDE, else the solution cannot be convergent, so it cannot be stable.

$\psi_t + c\psi_x = 0$ Char. form $\frac{d\psi}{dt} = 0$ on char: $\frac{dx}{dt} = c$, i.e.

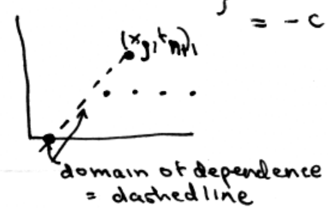
$\psi = f(x - ct)$ $x - x_j = c(t - t^{n+1})$

$= -c\Delta t$ at $t = t^n$

$\Rightarrow$ If FDA includes points $j-1, j$ at time n,

$x_{j-1} < x_j - c\,\Delta t < x_j$ for convergence

$\Rightarrow \quad 0 \le \mu \le 1 \quad , \mu = \frac{c\Delta t}{\Delta x}$



For a method including points $j-2, j-1, j, j+1$,
CFL condition is $\quad -2 < \mu < 1$

For a problem with multiple wave speeds, e.g. linearized 1D sound waves

$$u_t = -\frac{1}{\rho_0} p_x$$
$$\frac{1}{\rho_0 c^2} p_t = -u_x \qquad \Rightarrow p_{tt} - c^2 p_{xx} = 0$$

then both wavespeeds must satisfy CFL condition, so an FDA including points $j-1, j, j+1$ at time n must satisfy $\quad 0 < \mu < 1$

and ~~an~~ scheme using one side differences, $j-2, j-1, j$ will always be unstable.