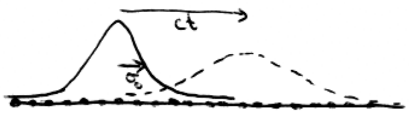
The modified eqn takes into account the leading error term in the num disp. relation. Concepts of phase and group velocity apply to the discrete dispersion relation just as they would for the exact dispersion relation, e.g. for upwind

$$\frac{\omega}{k} = \frac{\partial \omega}{\partial k} = c + O((\Delta x)^2)$$

$\Rightarrow$ nondispersive phase/energy propagation to this order.



$$\sigma^2 = \sigma_0^2 + 4Dt \quad , \quad D = \frac{c\Delta x}{2}(1-\mu) = \text{"numerical diffusion"}$$

Separation of Space and Time Differencing

Many (but not all) FDAs involve discretizations of space and time derivatives that can be separated, e.g.

$$\psi_t + c\psi_x = 0$$

Upwind:

$$\underbrace{\delta_t^F \phi_j^n}_{\frac{\partial}{\partial t}} + \underbrace{c\,\delta_x^B \phi_j^n}_{c\frac{\partial}{\partial x}} = 0$$

i.e. "Forward Euler" in t
"Backward Euler" in x
or "upwind difference"

Let's step back for a second and consider the time and space differencing issues separately. If the PDE can be written

$$\psi_t = S[\psi]$$

where S is the space-differentiation part of the PDE, and S is constant-coefficient, and S^{FDA} is a constant-coeff FDA to S, then we can write any function $\phi_j(t)$ on the discrete grid in x in terms of its Fourier components $\hat{\phi}^k(t)e^{ikx_j}$, so it suffices to analyze the time-differencing of

$$e^{ikx_j}\hat{\phi}^k_t = S^{FDA}\left[\hat{\phi}^k e^{ikx_j}\right] = \sigma^k e^{ikx_j}\hat{\phi} \quad , \text{ i.e. } \hat{\phi}^k_t = \sigma^k \hat{\phi}.$$

Thus, what we need ~~to~~ know is the behavior of our time-discretization on the amplification equation

$$\frac{d\psi}{dt} = \sigma\psi.$$

over all (complex) choices of σ and Δt. This is also a good model eqn for analyzing ODE time stepping methods. $\sigma = ik$ gives the oscillation equation

$$\frac{d\psi}{dt} = ik\psi$$

More sophisticated time differencing

Let us look at the variety of methods for solving $\frac{d\psi}{dt} = F(\psi)$, (F possible including space derivs.), which are useful for time-differencing in ODE's and PDEs.

Two-level methods

... use only ϕ^n ~~in the~~ computation of ϕ^{n+1}. If the only function evaluations $F(\cdot)$ required are at ϕ^n the method is a single-stage explicit method, i.e.g.

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = F(\phi_n) \qquad \text{(forward Euler scheme)} \;-1^{st}\text{ order accurate}$$

If function evaluations are also required at ϕ^{n+1} the method is a single-stage implicit method, e.g

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = F(\phi^{n+1}) \qquad \text{(backward Euler)} \;-1^{st}\text{ order acc.}$$

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \frac{F(\phi^{n+1})+F(\phi^n)}{2} \qquad (\text{trapezoidal} - 2^{nd}\text{ order acc}).$$

Note that now ϕ^{n+1} may have to be determined iteratively if $F(\cdot)$ isn't simple.

Analysis using amplification equation

Forward Euler

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \sigma\phi^n \;, \quad \phi^{n+1} = A(\sigma)\phi^n \;, \; A(\sigma) = 1+\sigma\Delta t$$

For $\sigma = i\kappa$, $|A(\sigma)| = [1+\kappa^2(\Delta t)^2]^{1/2} > 1 \Rightarrow$ unstable, and

$$R = \frac{\text{phase } A}{|\kappa\Delta t|} = \frac{\tan^{-1}\kappa\Delta t}{\kappa\Delta t} \approx 1 - \frac{(\kappa\Delta t)^2}{3}\ldots < 1 \text{ (decelerating)}$$

For σ = real, $A/A_{ex} = \frac{1+\sigma\Delta t}{e^{\sigma\Delta t}} = \begin{cases} >1 & \text{for } \sigma<0 \\ <1 & \text{for } \sigma>0 \end{cases}$

For $\sigma < -\frac{2}{\Delta t}$, $|A| > 1 \Rightarrow$ unstable, again.