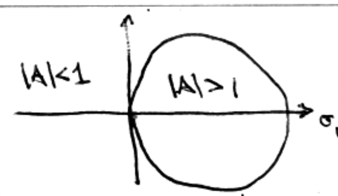
<u>Backward Euler</u> $A(\sigma) = \frac{1}{1-\sigma\Delta t}$



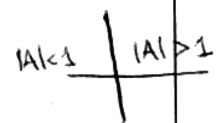
For $\sigma = i\kappa$, $|A(\sigma)| = [1+\kappa^2(\Delta t)^2]^{-1/2} < 1$ (damping)

$$R = \frac{\tan^{-1}\kappa\Delta t}{\kappa\Delta t} \text{ as before}$$

For σ real, $\frac{A}{A_{ex}} = \frac{1}{e^{\sigma\Delta t}(1-\sigma\Delta t)} = \begin{cases} <1 & \text{for } \sigma<0 \\ >1 & \text{for } \sigma>0 \end{cases}$

If $\sigma > \frac{1}{\Delta t}$, $\frac{A}{A_{ex}} < 0$ and we get a spurious oscillation, but for $\sigma < 0$, the scheme is well-behaved.

<u>Trapezoidal</u>

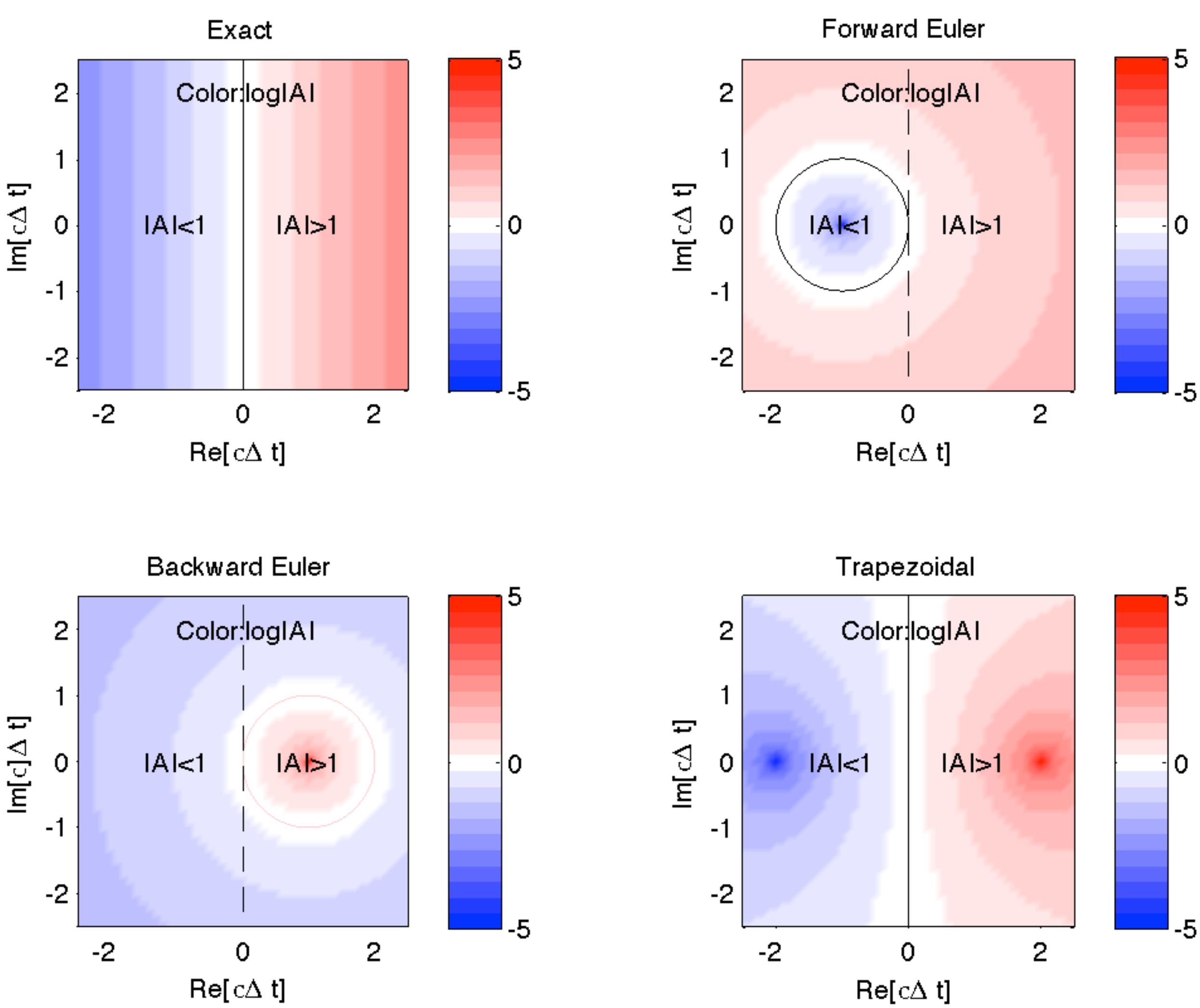


$$A(\sigma) = \frac{1+\frac{\sigma\Delta t}{2}}{1-\frac{\sigma\Delta t}{2}}$$

For $\sigma = i\kappa$, $|A(\kappa)| = 1$ and $\frac{\text{phase } A}{i\kappa\Delta t} = \frac{2\tan^{-1}\frac{\kappa\Delta t}{2}}{\kappa\Delta t} \approx 1 - \frac{(\kappa\Delta t)^2}{12}$ (weakly decelerating)

If $\sigma > \frac{2}{\Delta t}$, $\frac{A}{A_{ex}} < 0$ (spurious oscillation)

If Re $\sigma < 0$, $|A| < 1$ always, so spurious oscillation forbidden.



Stability regions (blue shades bounded by black contour) for Euler and trapezoidal methods

As an example of using implicit time differencing in a PDE for numerical stability, consider some methods for the diffusion equation $\psi_t = a\psi_{xx}$ using centered space differencing. In particular, we consider 3 methods of the form:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = a\left[c\delta_x^2\phi_j^n + (1-c)\delta_x^2\phi_j^{n+1}\right], \quad c = \begin{cases} 1, & \text{Forward Euler} \\ 0, & \text{Backward Euler} \\ 1/2, & \text{Trapezoidal} \end{cases}$$

in which the space derivative is respectively evaluated at time n (forward Euler), $n + 1$ (backward Euler) and using a trapezoidal time average.

Since each method is linear and constant-coefficient, its spatial eigenfunctions have the form $\phi_j \propto \exp(ikx_j)$. Substituting this into the centered space derivative, we find that

$$\delta_x^2\phi_j = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2} = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}\phi_j = -\frac{2(1-\cos k\Delta x)}{\Delta x^2}\phi_j$$

Hence, for this wavenumber, if we define

$$\sigma(k) = -\frac{2a(1-\cos k\Delta x)}{\Delta x^2}$$

then

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = \sigma(k)\left[c\phi_j^n + (1-c)\phi_j^{n+1}\right]$$

which is in the standard form of an amplification equation. Hence, if we use forward-Euler time differencing (the FTCS method), the stability limit on wavenumber k is

$$\sigma(k)\Delta t = -\frac{2a\Delta t(1-\cos k\Delta x)}{\Delta x^2} > -2$$

For this to apply for all wavenumbers k, for which $0 < 1 - \cos k\Delta x < 2$,

$$\sigma_{\max}\Delta t = -\frac{4a\Delta t}{\Delta x^2} > -2 \quad \Rightarrow \quad \nu = \frac{a\Delta t}{\Delta x^2} < \frac{1}{2} \quad \text{for stability of FTCS}$$

Similar reasoning implies the BECS and trapezoidally time-differenced methods are stable for all wavenumbers at all timesteps.

The BECS method requires solving the tridiagonal system

$$\left(1 - a\Delta t\delta_x^2\right)\phi_j^{n+1} = -\nu\phi_{j-1}^{n+1} + \phi_j^{n+1} - \nu\phi_{j+1}^{n+1} = \phi_j^n,$$

In 1 space dimension, the tridiagonal system takes O(N) operations (flops) for a grid of N points, hence is of comparable computational expense per timestep as FTCS. Because the method is 1st order accurate in time (due to the forward time difference) vs. 2nd order in space (due to the centered space difference) the truncation error $T = \alpha\Delta t + \beta\Delta x^2$. The most efficient tradeoff between space and time differencing (we'll show later) is when the space and time truncation errors are comparable, i. e. if $\nu = a\Delta t/\Delta x^2 = O(1)$. Thus even though BECS has no stability limit on Δt, its accuracy does become compromised for $\nu \gg 1$.

Using trapezoidal time differencing (the 'Crank-Nicolson' method), we again can efficiently solve a tridiagonal system. Now the the truncation error $T = \gamma\Delta t^2 + \beta\Delta x^2$ so efficiency mandates a much larger timestep $\Delta t = O(\Delta x)$, and numerical stability doesn't prevent us from doing this. Hence the C-N method is quite attractive and commonly used for diffusion problems.