Multistage two level schemes

... use additional function evaluations to make an accurate explicit scheme, e.g. a general two level.

$$\tilde{\phi}^{n+\alpha} = \phi^n + \alpha \Delta t\, F(\phi^n)$$

$$\phi^{n+1} = \phi^n + \beta \Delta t\, F(\tilde{\phi}^{n+\alpha}) + (1-\beta)\Delta t\, F(\phi^n)$$

We would usually like α, β to be chosen to get second-order accuracy.

If $F(\phi) = \sigma\phi$,

$$\tilde{\phi}^{n+\alpha} = (1 + \alpha\sigma\Delta t)\,\phi^n$$

$$\phi^{n+1} = (1 + (1-\beta)\Delta t \cdot \sigma)\,\phi^n + \sigma\beta\Delta t\, \tilde{\phi}^{n+\alpha}$$

$$= \Big(\underbrace{1 + (1-\beta)\Delta t\cdot\sigma + \sigma\beta\Delta t(1 + \alpha\sigma\Delta t)}_{A(\sigma)}\Big)\,\phi^n$$

$$A(\sigma) = 1 + \sigma\Delta t + \alpha\beta(\sigma\Delta t)^2$$

For 2^{nd} order accuracy, $\alpha\beta = \frac{1}{2}$. All such methods are Runge-Kutta methods:

$\alpha = 1,\ \beta = \frac{1}{2}$ ⇒ trapezoidal R-K

$\alpha = \frac{1}{2},\ \beta = 1$ ⇒ midpoint R-K.

2^{nd} –order Runge-Kutta stability analysis, for either 2^{nd}-order R-K method:

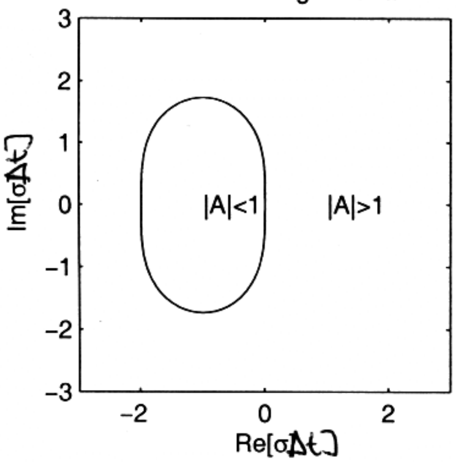
$A = 1 + (\sigma\Delta t) + (\sigma\Delta t)^2/2$

TimeDifferencingStabilityRegion.m (downloadable from class web page)

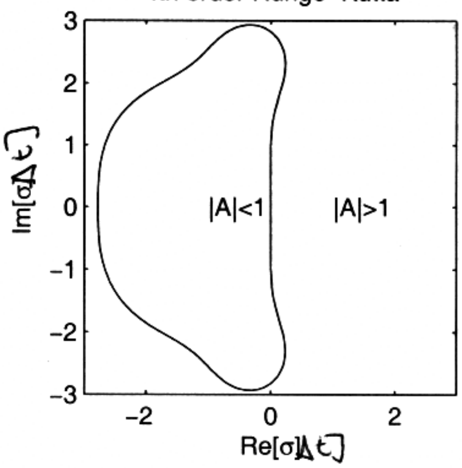
```
% s = sigma*dt

sr = -3:0.1:3;
si = -3:0.1:3;
[Sr Si] = meshgrid(sr,si);
S = Sr + Si*1i;

A = 1 + S + S.^2/2;  % Amplification factor
contour(sr,si,abs(A) ,[1,1]); % Plot |A|=1 contour
```

Absolute stability regions

2nd order Runge–Kutta



4th order Runge–Kutta



$\Rightarrow$ If σ = real need $-\sigma\Delta t < 2$ for absolute stability (like Forward Euler)

If σ real, need $-\sigma\Delta t \lesssim 2.8$ for absolute stability. If σ imag $= ik$ need $|k\Delta t| \lesssim 2.8$ for stability.

4th order Runge-Kutta; a 2-level, 4-stage, 4th-order accurate method.

$$q_1 = \Delta t\, F(\phi^n)$$
$$q_2 = \Delta t\, F(\phi^n + q_1/2)$$
$$q_3 = \Delta t\, F(\phi^n + q_2/2)$$
$$q_4 = \Delta t\, F(\phi^n + q_3)$$
$$\phi^{n+1} = \phi^n + \frac{1}{6}\{q_1 + 2q_2 + 2q_3 + q_4\}$$

Stability analysis: $F(\phi) = \sigma\phi$

$$q_1 = \sigma\Delta t\,\phi^n$$
$$q_2 = \sigma\Delta t\left(\phi^n + \frac{q_1}{2}\right) = \sigma\Delta t\left\{1 + \frac{\sigma\Delta t}{2}\right\}\phi^n$$
$$q_3 = \sigma\Delta t\left(\phi^n + \frac{q_2}{2}\right) = \sigma\Delta t\left\{1 + \frac{\sigma\Delta t}{2} + \frac{(\sigma\Delta t)^2}{4}\right\}\phi^n$$
$$q_4 = \sigma\Delta t\,(\phi^n + q_3) = \sigma\Delta t\left\{1 + \sigma\Delta t + \frac{(\sigma\Delta t)^2}{2} + \frac{(\sigma\Delta t)^3}{4}\right\}\phi^n$$
$$\frac{\phi^{n+1}}{\phi^n} = A(\sigma) = 1 + \frac{\sigma\Delta t}{6}\left\{1 + 2\left(1 + \frac{\sigma\Delta t}{2}\right) + 2\left(1 + \frac{\sigma\Delta t}{2} + \frac{(\sigma\Delta t)^2}{4}\right) + 1 + \sigma\Delta t + \frac{(\sigma\Delta t)^2}{2} + \frac{(\sigma\Delta t)^3}{4}\right\}$$
$$= 1 + \sigma\Delta t + \frac{1}{2}(\sigma\Delta t)^2 + \frac{1}{6}(\sigma\Delta t)^3 + \frac{1}{24}(\sigma\Delta t)^4 \qquad [= e^{\sigma\Delta t} + O((\sigma\Delta t)^5)]$$

in accord with 4th order accuracy.

The stable region includes oscillations with frequencies $|\sigma_i\Delta t| < 2.8$ and pure exponentially decaying modes with $-\sigma_r\Delta t < 2.8$. The favorable stability properties for oscillations and the high order of accuracy make the RK4 method especially popular for PDEs in which accurate wave propagation is important.