Atmospheric Science 441, Homework 7

1. Suppose the velocity in a patch of fluid is circularly symmetric and given as a function of the radial distance r from the center by the following:

$$V(r) = \begin{cases} Ar/r_0 & \text{if } r \le r_0, \\ Ar_0/r & \text{if } r > r_0. \end{cases}$$

where r_0 is a constant reference radius.

(a) Derive an expression for the vorticity of the fluid in the region $r \le r_0$.

(b) Derive an expression for the vorticity of the fluid in the region $r > r_0$.

2. Starting from the horizontal momentum equations in z-coordinates,

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

we found that the vertical vorticity equation took the form

$$\frac{D(\zeta + f)}{Dt} = \ldots + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right).$$

Here the "…" include the stretching and the tilting terms. The final term on the right is the "baroclinic" or solenoidal term.

Show that the baroclinic term is not present if we derive the corresponding relation for the change in the vertical vorticity component in isobaric coordinates. That is, show that starting from

$$\frac{Du}{Dt} - fv = -\left(\frac{\partial \Phi}{\partial x}\right)_p, \qquad \frac{Dv}{Dt} + fu = -\left(\frac{\partial \Phi}{\partial y}\right)_p,$$

the accelerations arising from height gradients never change the vertical vorticity on a constant pressure surface

$$\zeta_p = \left(\frac{\partial v}{\partial x}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p.$$

Hint: Do not bother deriving all the terms in the isobaric vertical vorticity equation, just focus on those related to pressure/height gradients and leave the others grouped as "…".

3. Suppose that, over the square domain $0 \leq x \leq L$, $0 \leq y \leq L$, all fields are periodic in both x and y. This means that for any field such as the u component of the velocity, and for fixed y_0 in the interval $[0, L]$, $u(0, y_0) = u(L, y_0)$ and for any fixed x_0 in the interval $[0, L]$, $u(x_0, 0) = u(x_0, L)$. Show that the integral of ζ over this domain is zero.

[*Using more complex mathematics, this result generalizes to global averages of isentropic vorticity on an isentropic surface as in problem 4.16).*]