Dynamics of a zonally symmetric vortex

Cartesian geometry
f -plane
zonal wind is geostrophic
meridional wind is ageostrophic
eddy fluxes, diabatic heating, and friction are specified

We will be able to determine

MMC (diagnostic)
future evolution of wind and temperature fields

The governing equations

Three equations
Three dependent variables
Boundary conditions
Initial conditions

$$\frac{\partial u}{\partial t} = -f\frac{\partial \psi}{\partial p} + G + F$$

$$\frac{\partial \alpha}{\partial t} = \sigma\frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial u}{\partial p} = \frac{1}{f}\frac{\partial \alpha}{\partial y}$$

temperature is replaced by specific volume

MMC are represented by a stream function

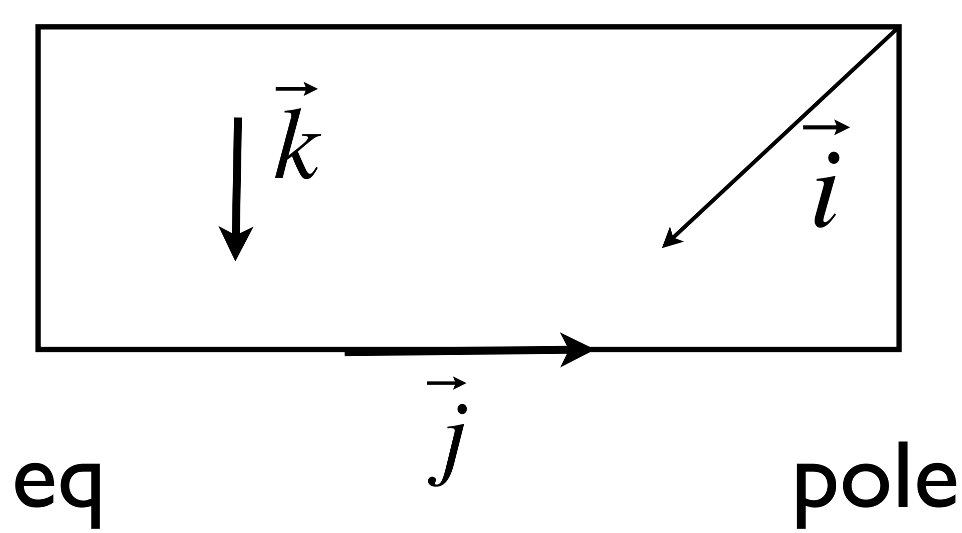
the pressure gradient force (PGF) in the meridional plane

$$\vec{\Sigma} = \left(\frac{\partial \widehat{\Phi}}{\partial y} \vec{j},\ \frac{\partial \widehat{\Phi}}{\partial p} \vec{k} \right) = \left(fu\ \vec{j},\ \widehat{\alpha}\ \vec{k} \right)$$

geostrophic equation hypsometric equation

$$\vec{i} \cdot \left(\nabla \times \vec{\Sigma} \right) = 0$$

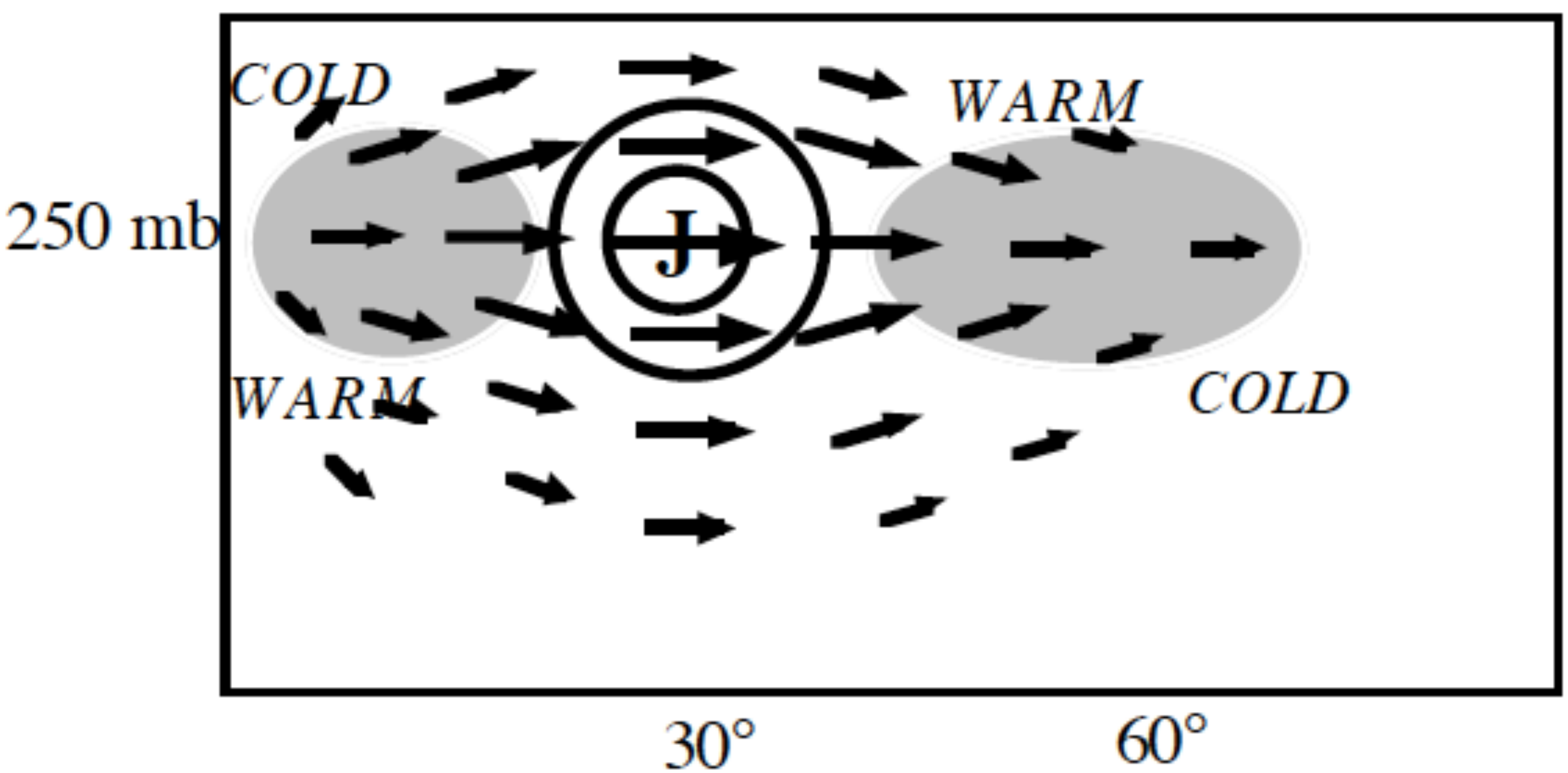
thermal wind equation:
the PGF is irrotational



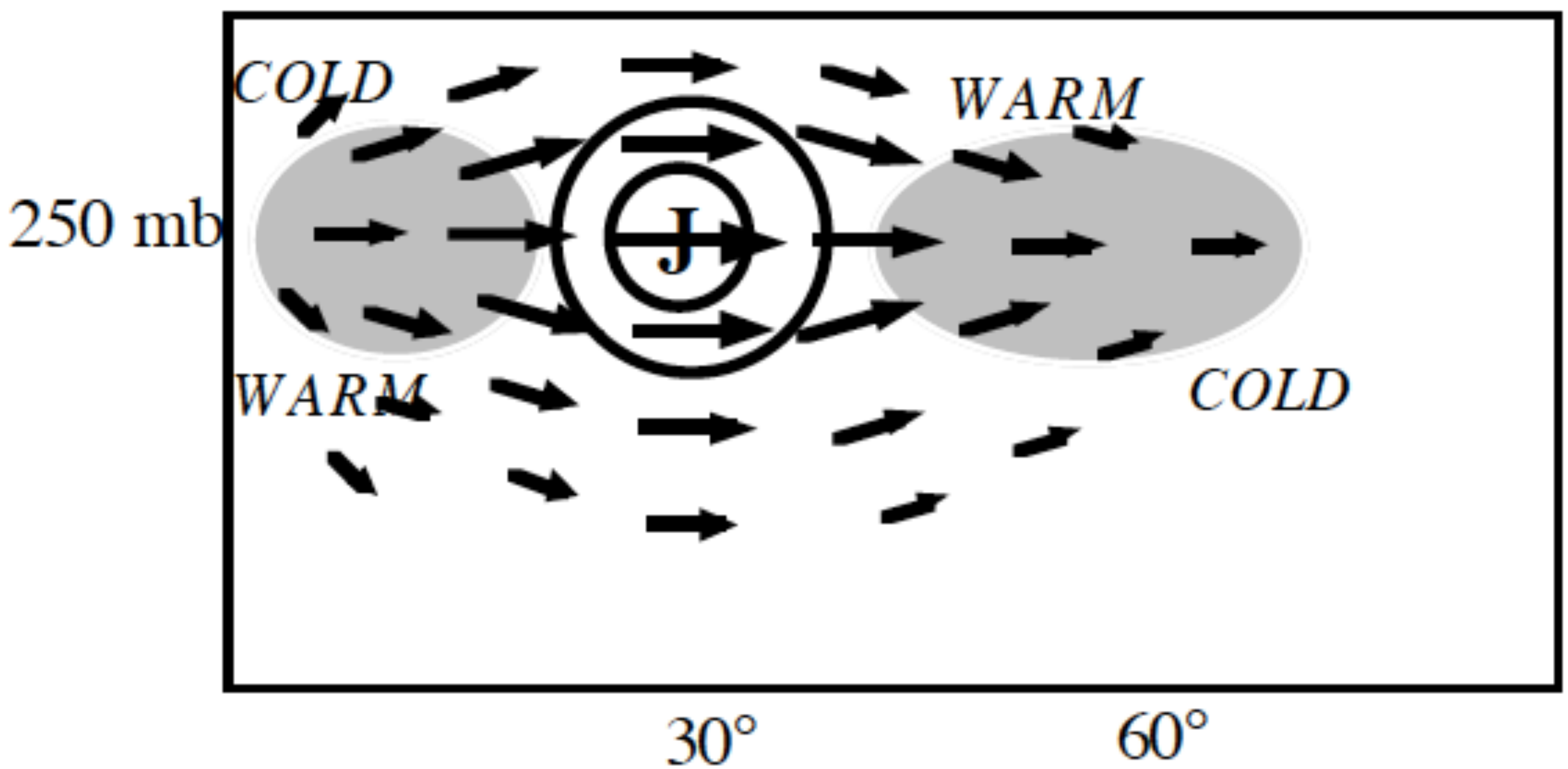
Concept of a stretched membrane

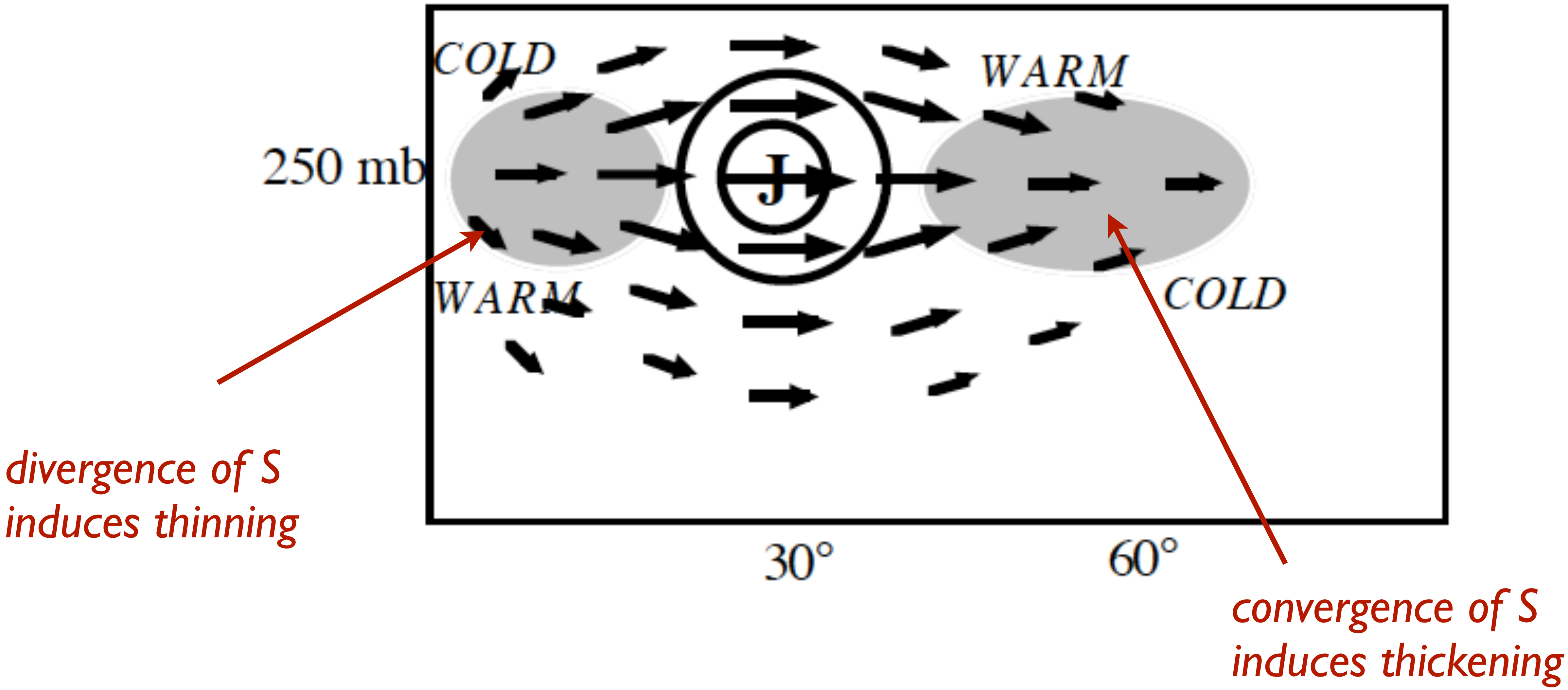
$$\vec{S} \equiv \left(\frac{u}{f}\,\vec{j},\ \frac{\widehat{\alpha}}{\sigma}\,\vec{k} \right)$$

the displacement vector



It is helpful to think of the displacement vector as relating to a stretched membrane. If the flow were at rest so that the displacement were everywhere equal to zero the thickness of the membrane would be uniform. Displacements act to thicken the membrane in some areas and thin it in others.

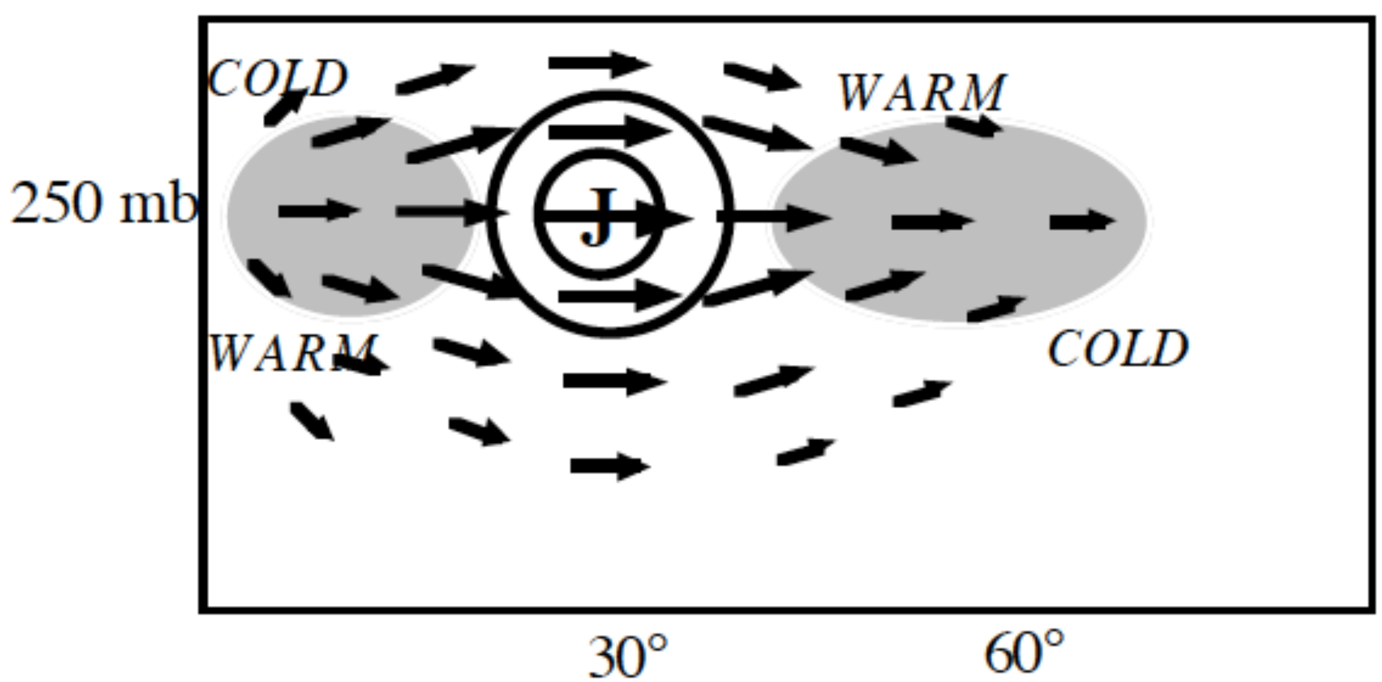



COLD
WARM
250 mb
J
WARM
COLD
30°
60°
divergence of S
induces thinning
convergence of S
induces thickening

$$\vec{S} \equiv \left(\frac{u}{f}\vec{j},\ \frac{\widehat{\alpha}}{\sigma}\vec{k} \right) \qquad -\nabla \cdot S = -\frac{1}{f}\frac{\partial u}{\partial y} - \frac{\partial}{\partial p}\frac{\widehat{\alpha}}{\sigma}$$

$$-f\nabla \cdot S = -\frac{\partial u}{\partial y} - f\frac{\partial}{\partial p}\frac{\widehat{\alpha}}{\sigma}$$

quasi-geostrophic potential vorticity



so membrane mass is a measure of potential vorticity!

Modulus of elasticity

$$\vec{\Sigma} = fu\,\vec{j}, \hat{\alpha}\,\vec{k}$$

$$\vec{S} \equiv \left(\frac{u}{f}\vec{j}, \frac{\hat{\alpha}}{\sigma}\vec{k} \right)$$

$$\vec{\Sigma} = \left(f^2, \sigma \right)\vec{S}$$

"stiffness" of membrane with respect to horizontal and vertical displacements

$$\frac{1}{2}\vec{\Sigma}\cdot\vec{S} = \frac{u^2}{2} + \frac{\hat{\alpha}^2}{2\sigma} \qquad \frac{1}{2g}\int_0^{p_0} \overline{\vec{\Sigma}\cdot\vec{S}}\,dp = \overline{K} + \overline{A}$$

Additional definitions

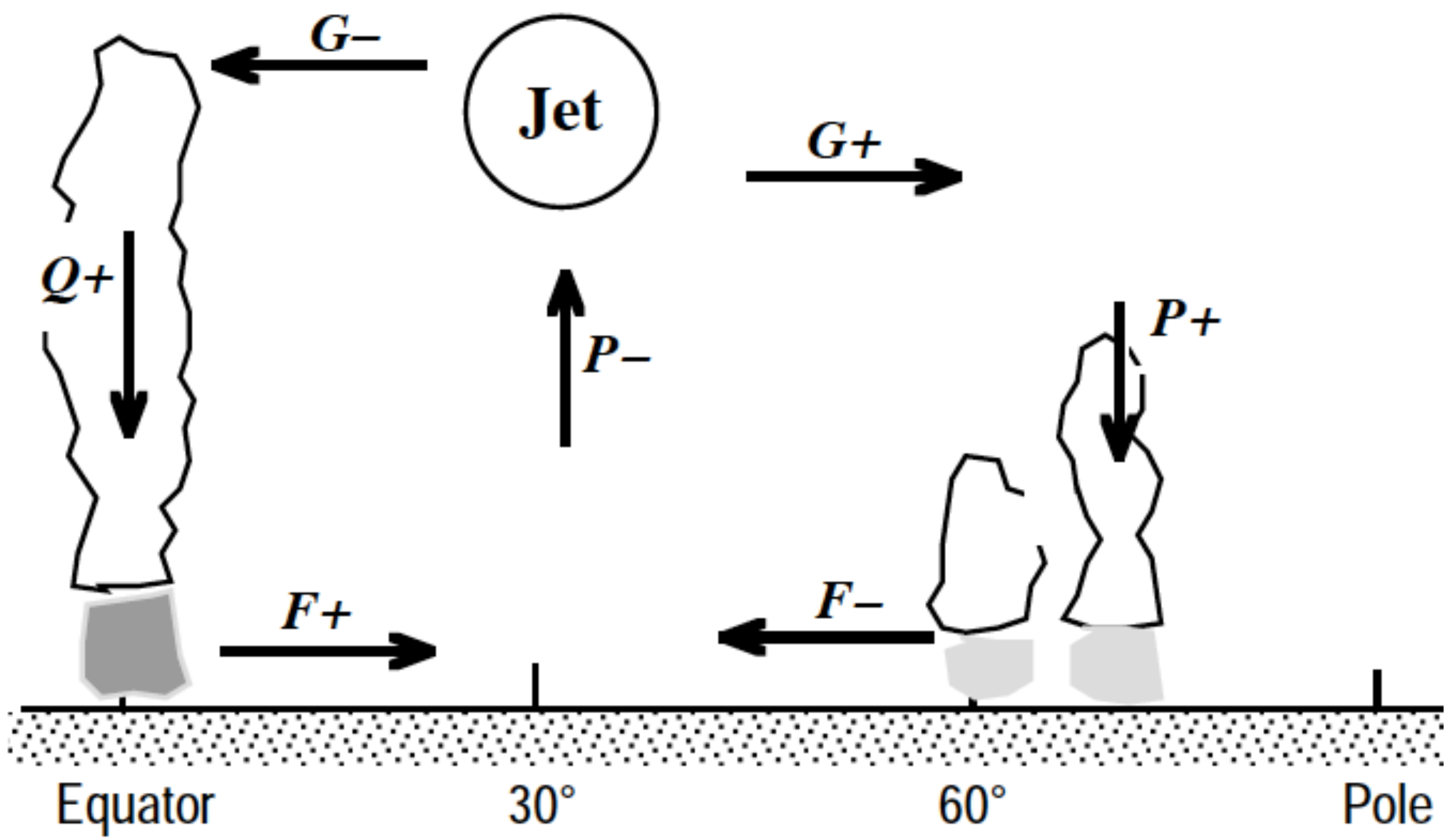
the MMC vector $\vec{\Psi} \equiv \vec{i} \times \nabla \psi = \left(v \vec{j}, \omega \vec{k} \right)$

the forcing vector $\vec{\Gamma} = \left(\frac{G+F}{f} \vec{j}, \frac{P+Q}{\sigma} \vec{k} \right)$

The prognostic equation

$$\frac{\partial \vec{S}}{\partial t} = \vec{\Gamma} + \vec{\Psi}$$

In a steady state; i.e., for the climatological-mean flow



for steady state $\vec{\Psi} = -\vec{\Gamma}$

from which we can deduce the existence of the Hadley and Ferrell cells

For the time-varying case, we eliminate the *d/dt* terms from the governing equations and solve for the MMC.

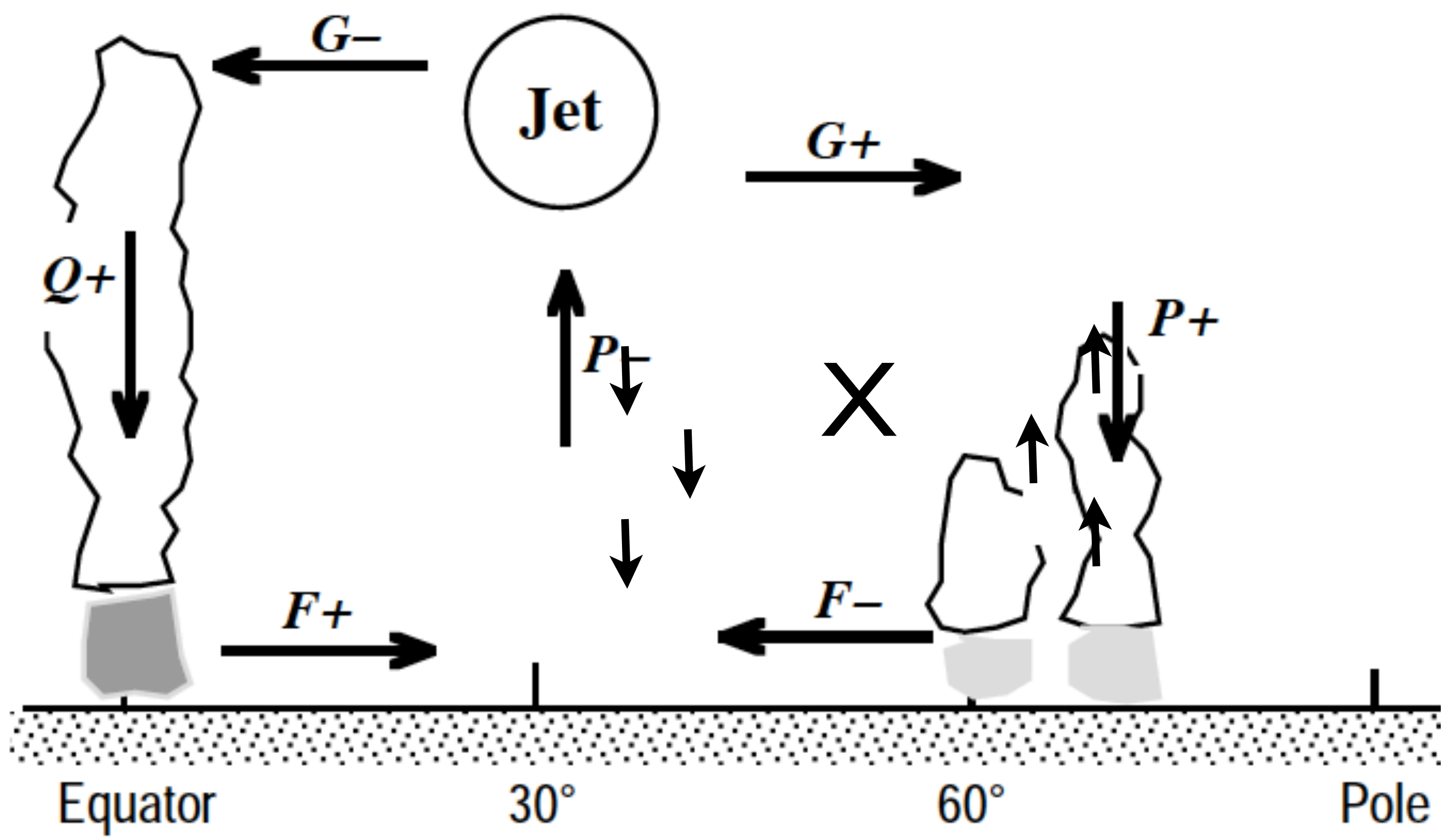
$$\frac{\partial u}{\partial t} = -f\frac{\partial \psi}{\partial p} + G + F$$

$$\frac{\partial \alpha}{\partial t} = \sigma\frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial u}{\partial p} = \frac{1}{f}\frac{\partial \alpha}{\partial y}$$

which yields

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$



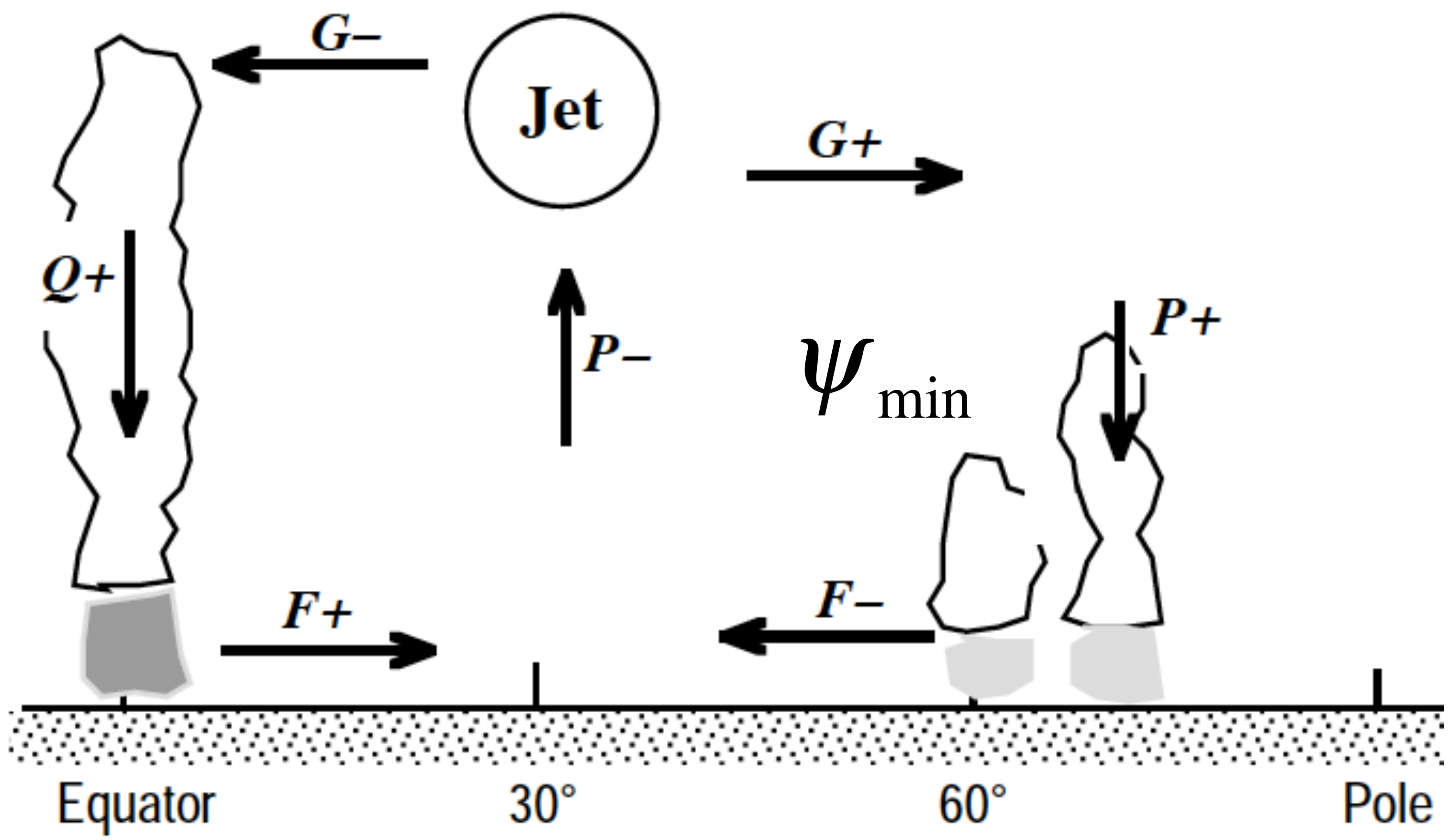
For the Ferrell cell in steady state:

at Point X both terms on the RHS exhibit maxima; i.e.,

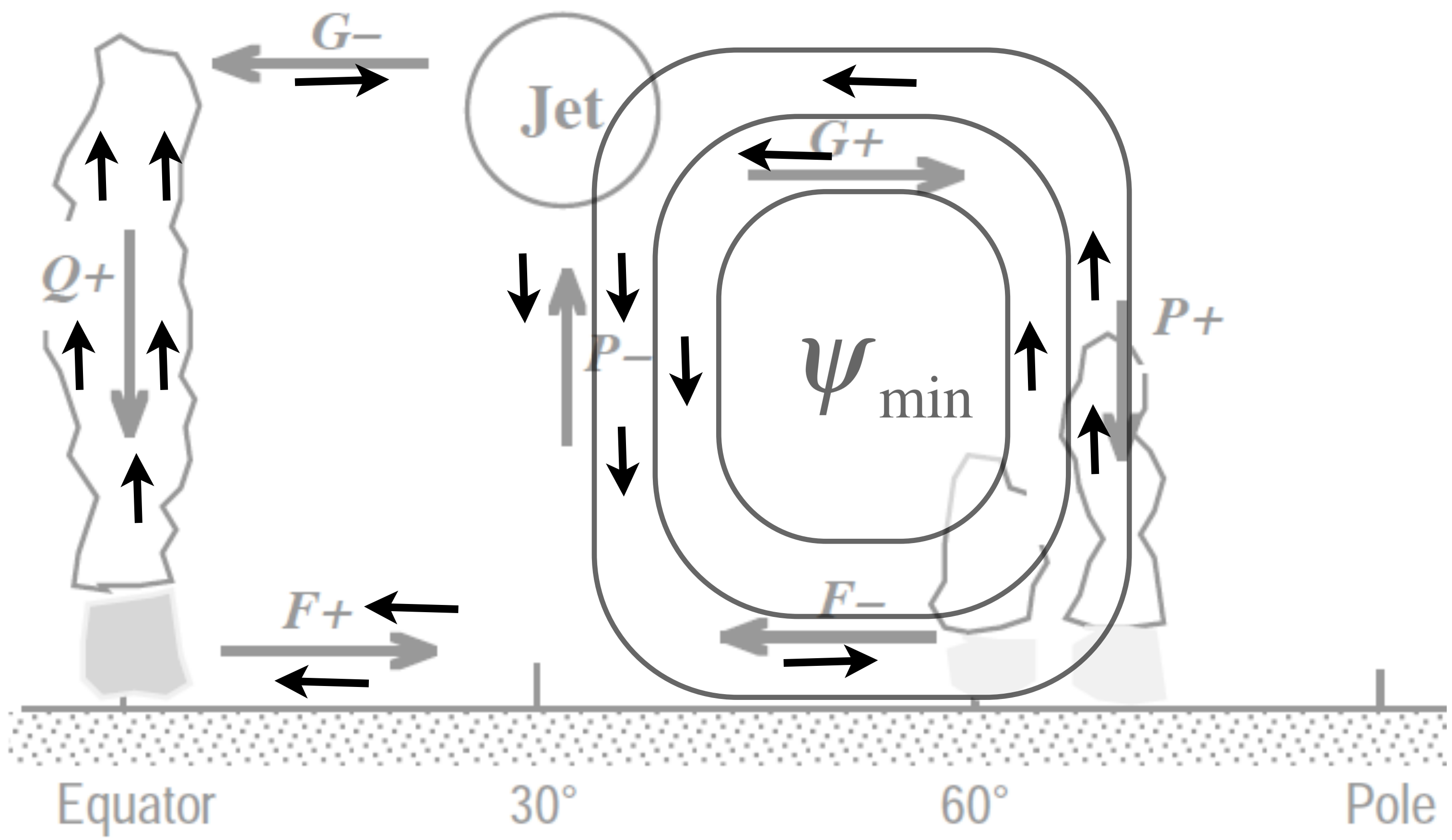
$G + F$ decreases with pressure

$P + Q$ increases with latitude

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

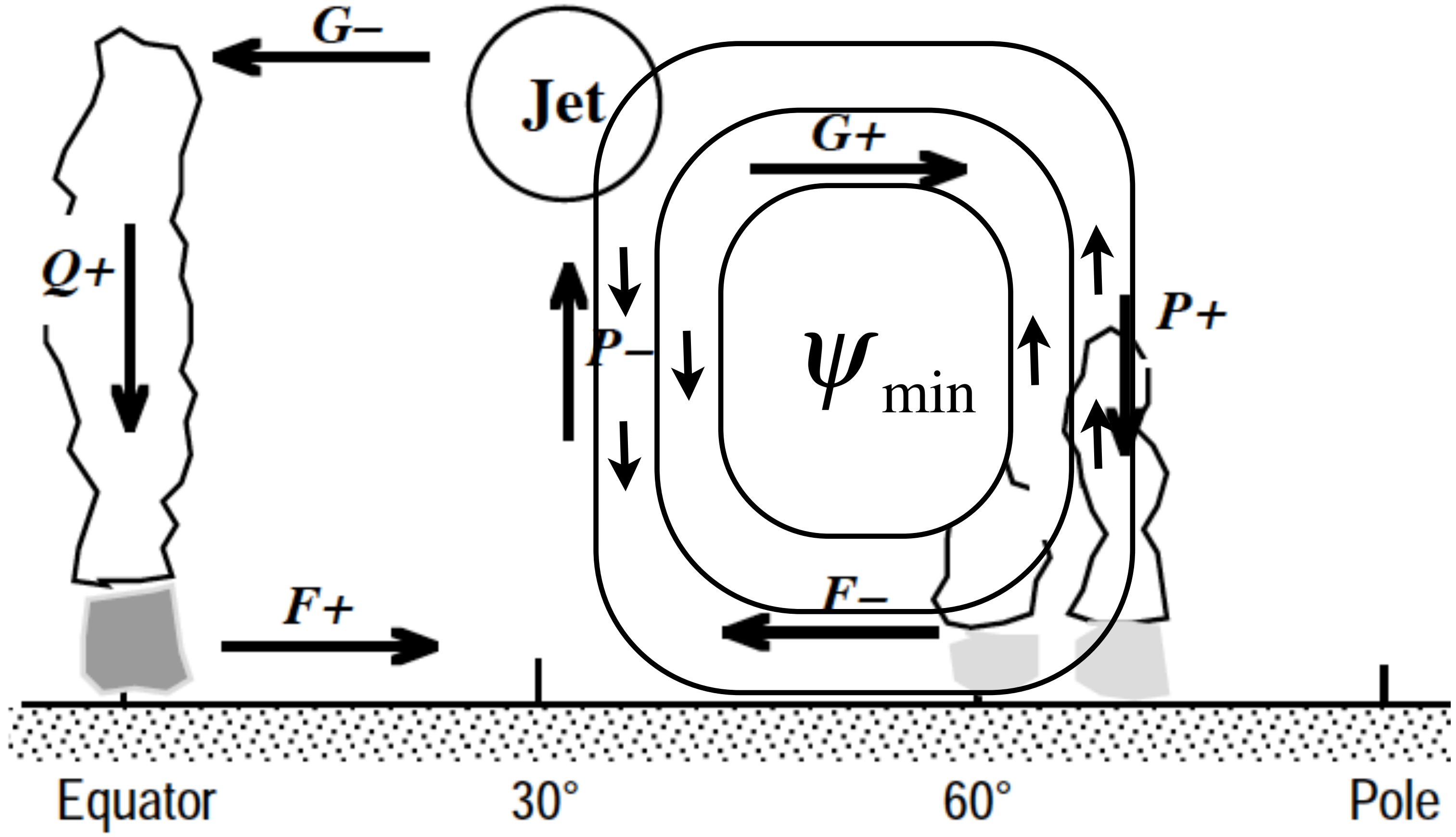


This is an elliptic equation, so where $A(\psi)$ exhibits a maximum, ψ exhibits a minimum.

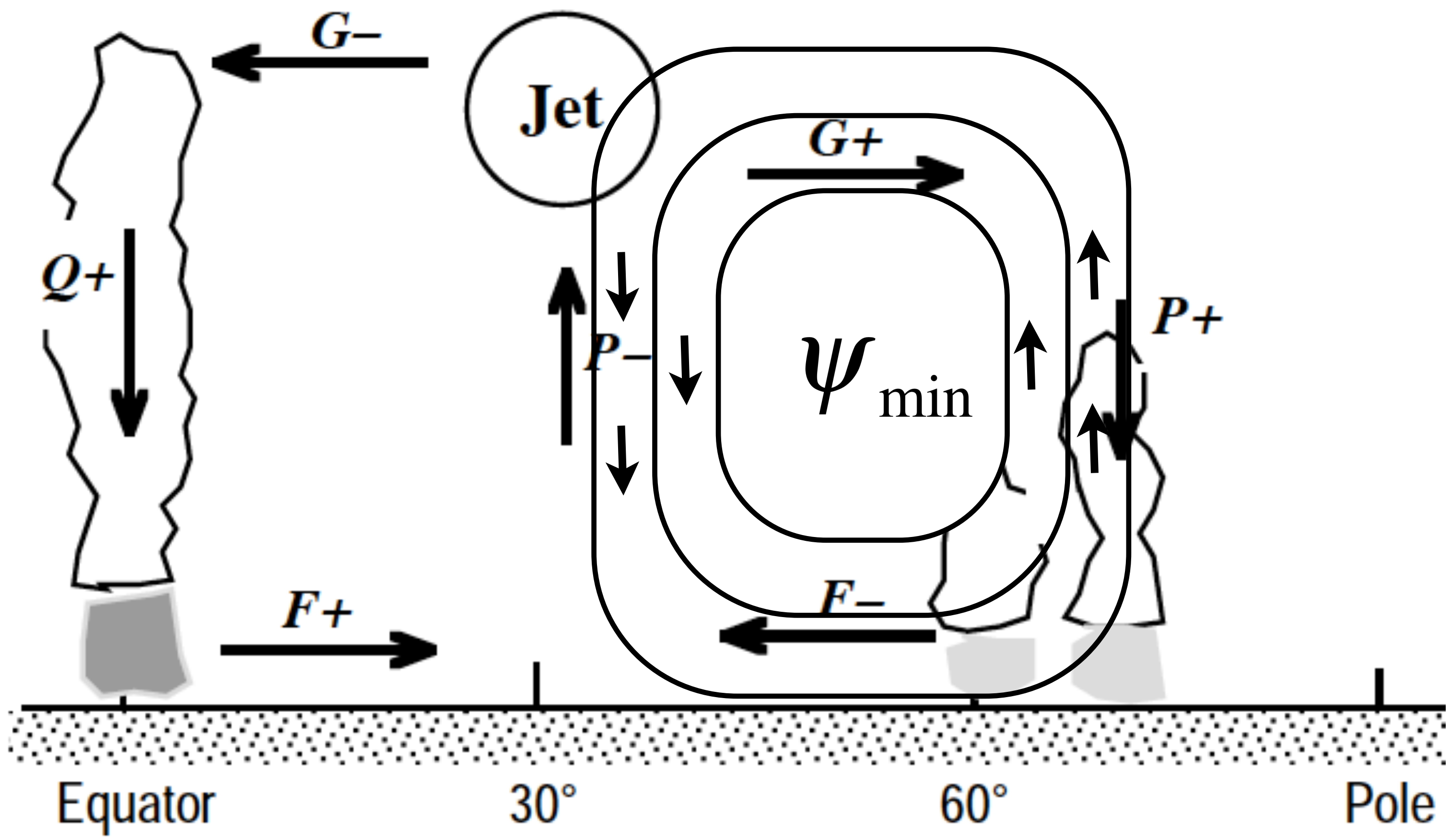


By convention $\vec{\Psi} \equiv \vec{i} \times \nabla \psi = \left(v\vec{j}, \omega\vec{k} \right)$

so the MMC circulate counterclockwise around $\psi_{\min}$



Note that with the vectorial notation we can sometimes infer the sense of the MMC without considering consider the elliptic equation. In a case like this, where the curl of the forcing is obvious, the sense of the MMC is also obvious.



For the time-mean MMC, *d/dt* = 0 but the solution for ψ is generally valid.

There are four ways of inferring the MMC

1. direct measurement of $[v]$

2. vorticity balance $[\overline{v}] = -\dfrac{G+F}{f}$

3. total energy balance $[\overline{\omega}] = -\dfrac{P+Q}{\sigma}$

4. eliminating time derivatives in governing equations

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

Four ways of inferring the MMC

1. direct measurement of $[v]$ — it's a small residual

2. vorticity balance $[\overline{v}] = -\dfrac{G+F}{f}$ — doesn't address the time dependence

3. total energy balance $[\overline{\omega}] = -\dfrac{P+Q}{\sigma}$ — doesn't address the time dependence

4. eliminating time derivatives in governing equations

$$A(\psi) \equiv \sigma \frac{\partial^2 \psi}{\partial y^2} + f^2 \frac{\partial^2 \psi}{\partial p^2} = -f \frac{\partial}{\partial p}(G+F) + \frac{\partial}{\partial y}(P+Q)$$

assumes geostrophy

There are four analogous ways of inferring ω in QG system

1. direct measurement of $\nabla \cdot \vec{V}$ it's a small residual

2. vorticity balance $\nabla \cdot \vec{V} = -\dfrac{\dfrac{\partial \varsigma}{\partial t} + \vec{V} \cdot \nabla \varsigma}{f + \varsigma}$ doesn't address the time dependence

3. total energy balance $\omega = -\dfrac{\dfrac{\partial T}{\partial t} + \vec{V} \cdot \nabla T}{\sigma}$ doesn't address the time dependence

4. eliminating time derivatives in governing equations

the omega equation assumes geostrophy

Vectorial interpretation of the elliptic equation for the MMC.

The zonal wind and thickness fields need to stay in thermal wind balance so

$$\frac{d}{dt}\left(\vec{i} \times \nabla\vec{\Sigma}\right) = 0$$

It follows that the curl of the tendency in $\vec{\Sigma}$ induced by the forcing vector must be balanced by the curl of the tendency induced by the MMC.

$\vec{\Sigma}$ can't be twisted because it's the gradient of a scalar

For 2D (zonally symmetric) flow, the omega equation is the requirement that the zonal wind and temperature fields remain in thermal wind balance as they evolve. The MMC inferred from this equation ensure that they stay in thermal wind balance.

Once we've solved for the MMC we have everything we need to solve the prognostic equations

$$\frac{\partial u}{\partial t} = -f\frac{\partial \psi}{\partial p} + G + F$$

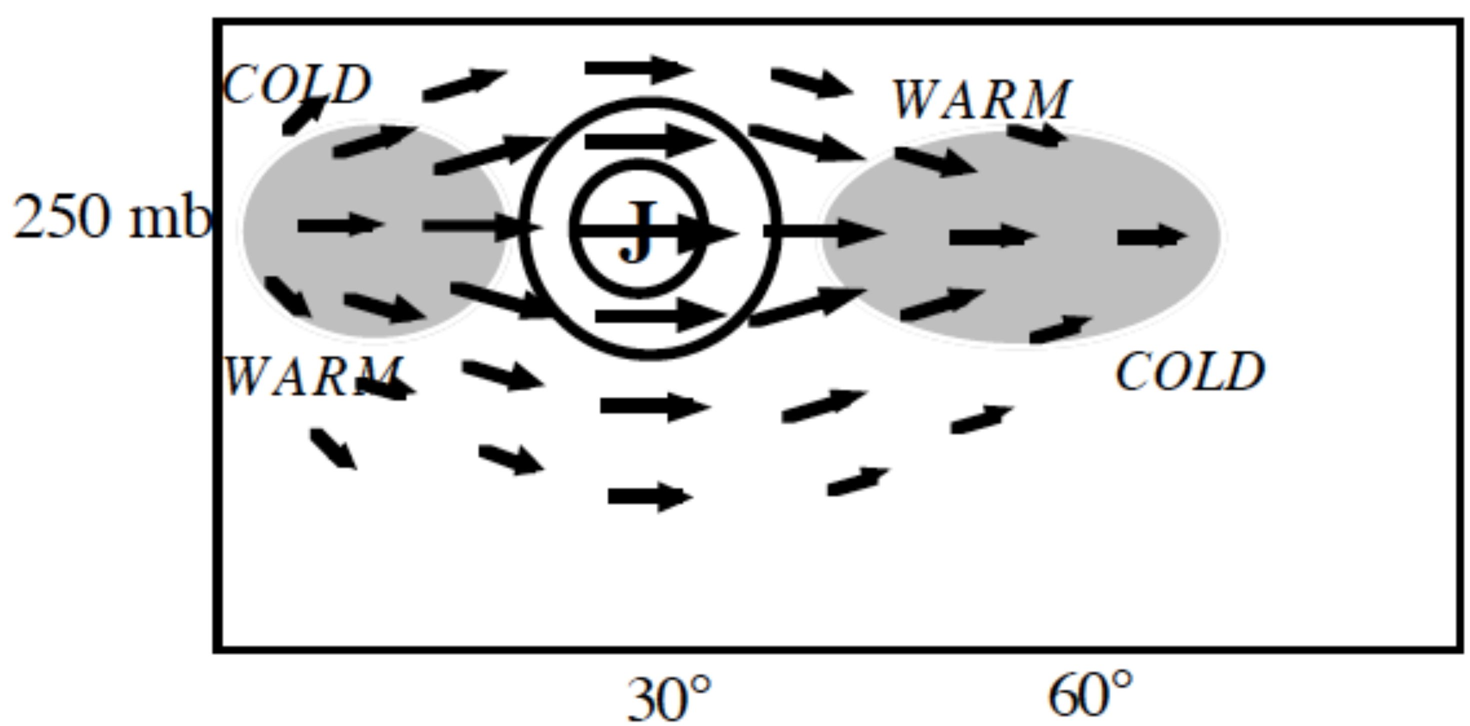
$$\frac{\partial \alpha}{\partial t} = \sigma\frac{\partial \psi}{\partial y} + P + Q$$

$$\frac{\partial \vec{S}}{\partial t} = \vec{\Gamma} + \vec{\Psi}$$

We can get insight into how the zonal flow evolves in response to the fluxes of heat and momentum by considering how the eddy fluxes of zonal momentum and temperature change the field of quasi-geostrophic potential vorticity q.

If we know how q is changing in response to the forcing, we can recover the fields of u, T, and Φ.

This is the so-called *invertibility principle*.



Recall that

$$q = -f\nabla \cdot \vec{S} = -\frac{\partial u}{\partial y} - f\frac{\partial}{\partial p}\frac{\widehat{\alpha}}{\sigma}$$

and note that the MMC are nondivergent and therefore do not have any effect on potential vorticity or membrane mass.

It follows that

$$\frac{dq}{dt} = \frac{d}{dt}\left(-f\nabla \cdot \vec{S}\right) = -\nabla \cdot \vec{\Gamma} = -\frac{\partial}{\partial y}\frac{(G+F)}{f} - \frac{\partial}{\partial p}\frac{(P+Q)}{\sigma}$$

Interpretation of the forcing terms

$$\frac{dq}{dt} = -\frac{\partial}{\partial y}\frac{(G+F)}{f} - \frac{\partial}{\partial p}\frac{(P+Q)}{\sigma}$$

vorticity forcing	static stability forcing
meridional pinching	vertical pinching

It's simple. Held (*JAS 1975*) proved that G can also be interpreted as the poleward eddy flux of vorticity; i.e.,

$$G \equiv -\frac{\partial}{\partial y}[u^* v^*] = [\varsigma^* v^*]$$

Proof: Start with the expression for *G* in either of its forms and transform it into the other form. Following Held (1975), assume that the flow in the eddies is nondivergent; i.e., that

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0$$

$G = \left[\varsigma * v * \right]$ That's awesome!
Is there an analogous expression for the poleward flux of potential vorticity $\left[q * v * \right]$?

As a matter of fact, there is.
Allow me to explain.

$$[q^* v^*] = \left[\varsigma^* - f \frac{\partial}{\partial p} \frac{\alpha^*}{\sigma} \right] v^* = G - f \frac{\partial}{\partial p} \frac{[v^* \alpha^*]}{\sigma}$$

$$\frac{\partial}{\partial p}[v^* \alpha^*] = \left[v^* \frac{\partial \alpha^*}{\partial p} \right] + \left[\alpha^* \frac{\partial v^*}{\partial p} \right]$$

If we approximate v^* by v_g^*, and use the thermal wind equation, the second term on the right hand-side vanishes.

$$-\left[v^* \frac{\partial \alpha^*}{\partial p} \right]$$

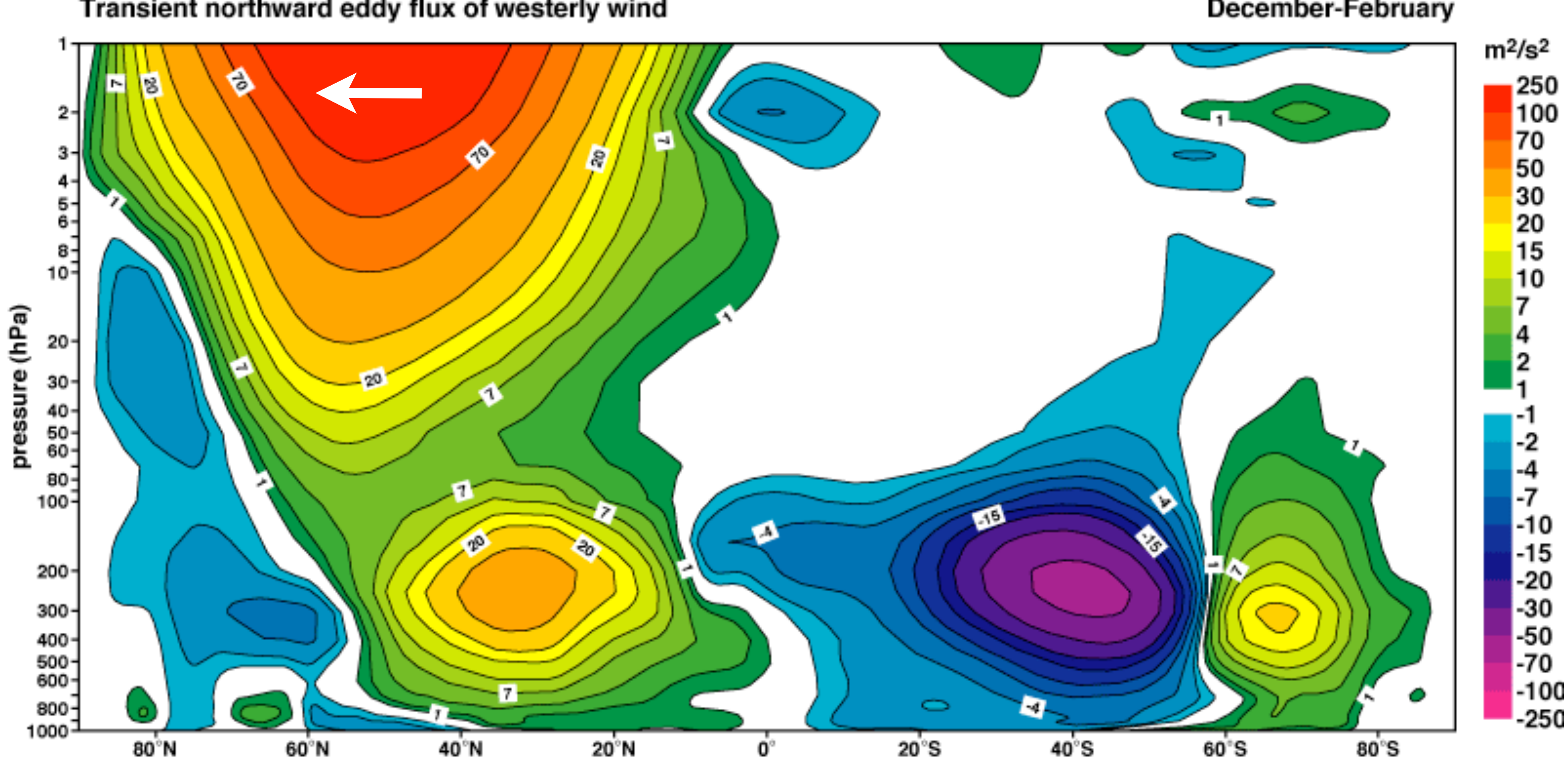
can be interpreted as
the poleward eddy flux of static stability

So to summarize,

$$\left[q^{*}v^{*}\right]=\left[\varsigma^{*}v^{*}\right]-f\frac{\partial}{\partial p}\frac{\left[v^{*}\alpha^{*}\right]}{\sigma}$$

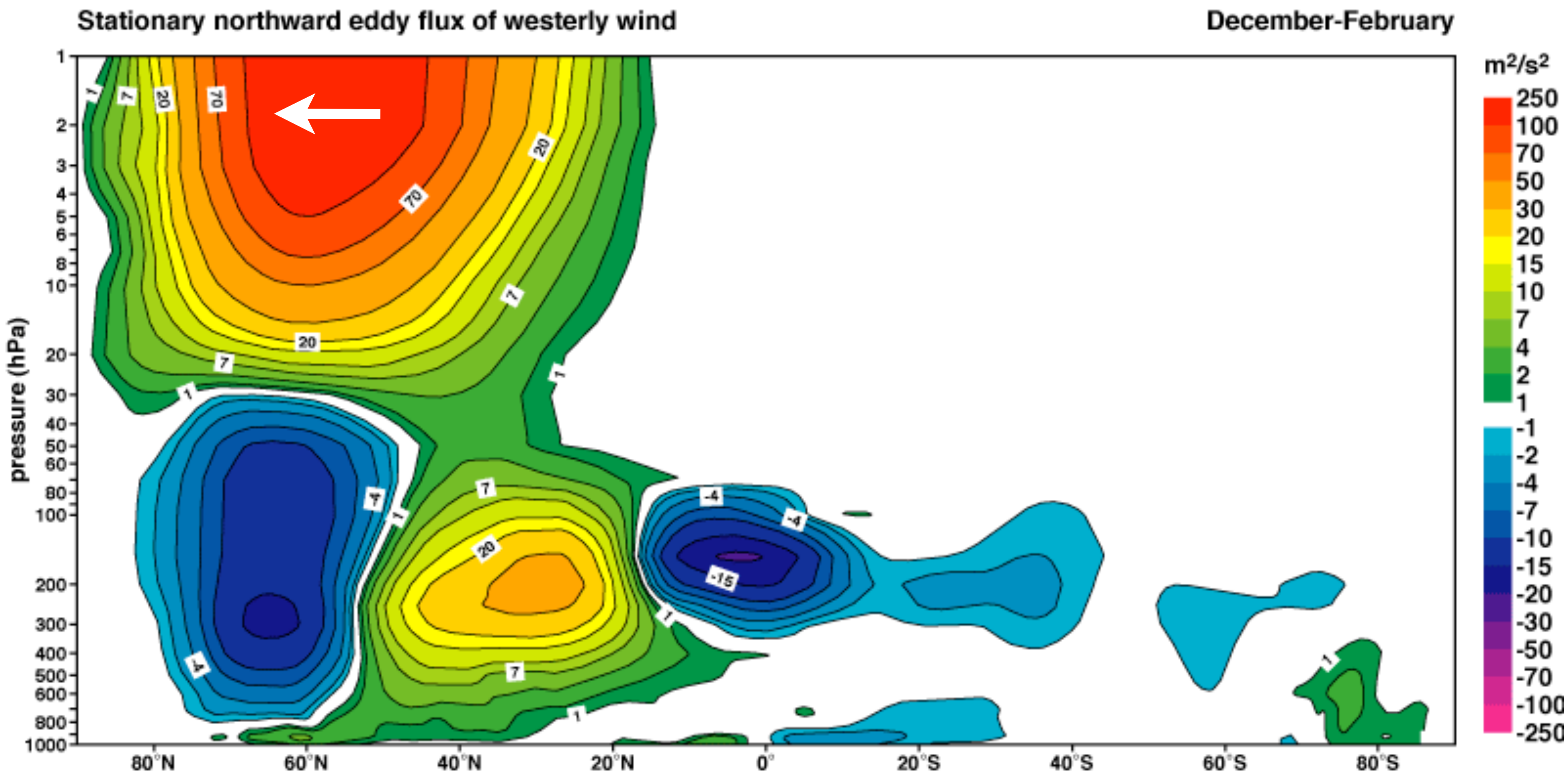
vorticity flux   static stability flux

An application

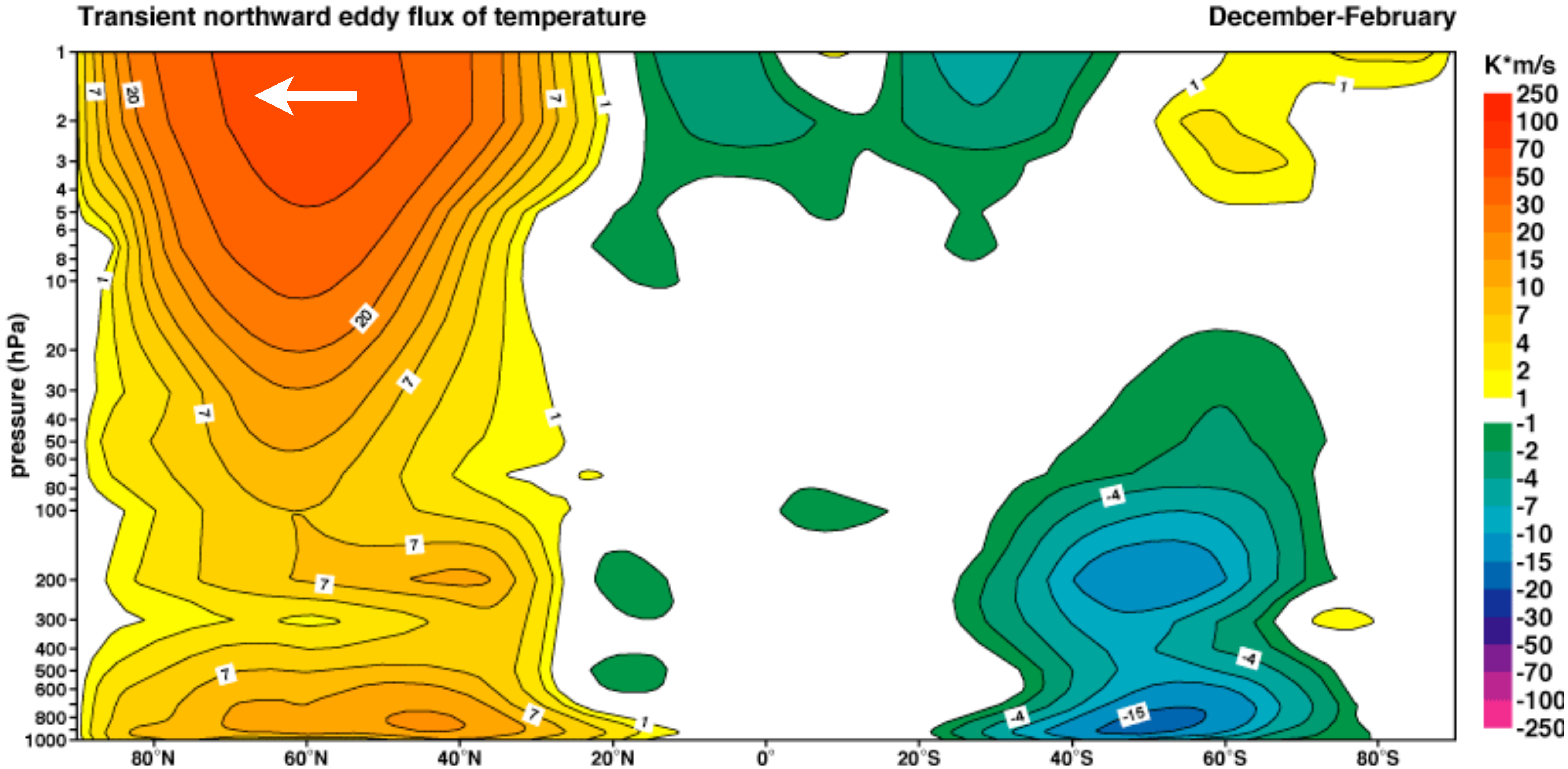


transient eddies

$$[u^* v^*]$$

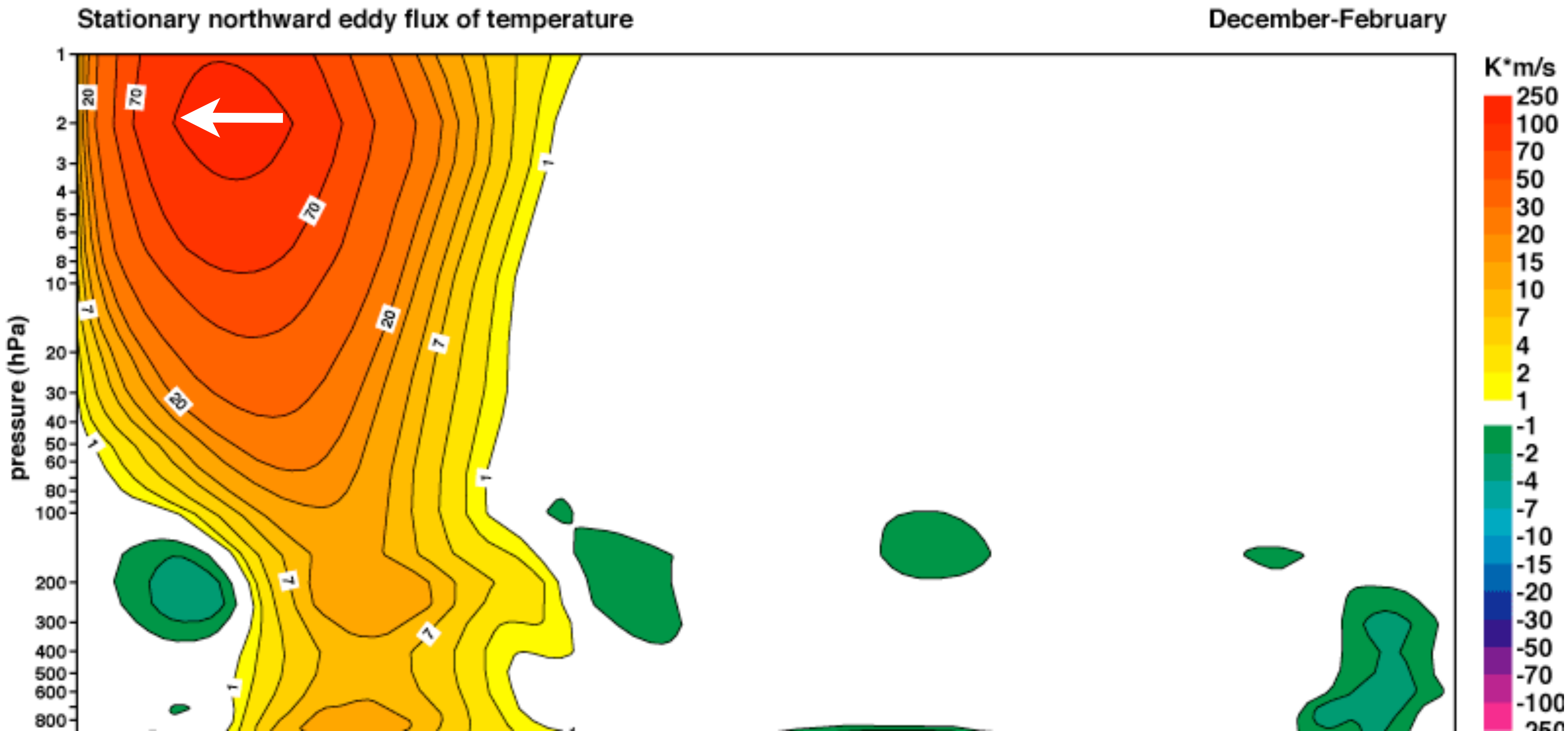


stationary waves

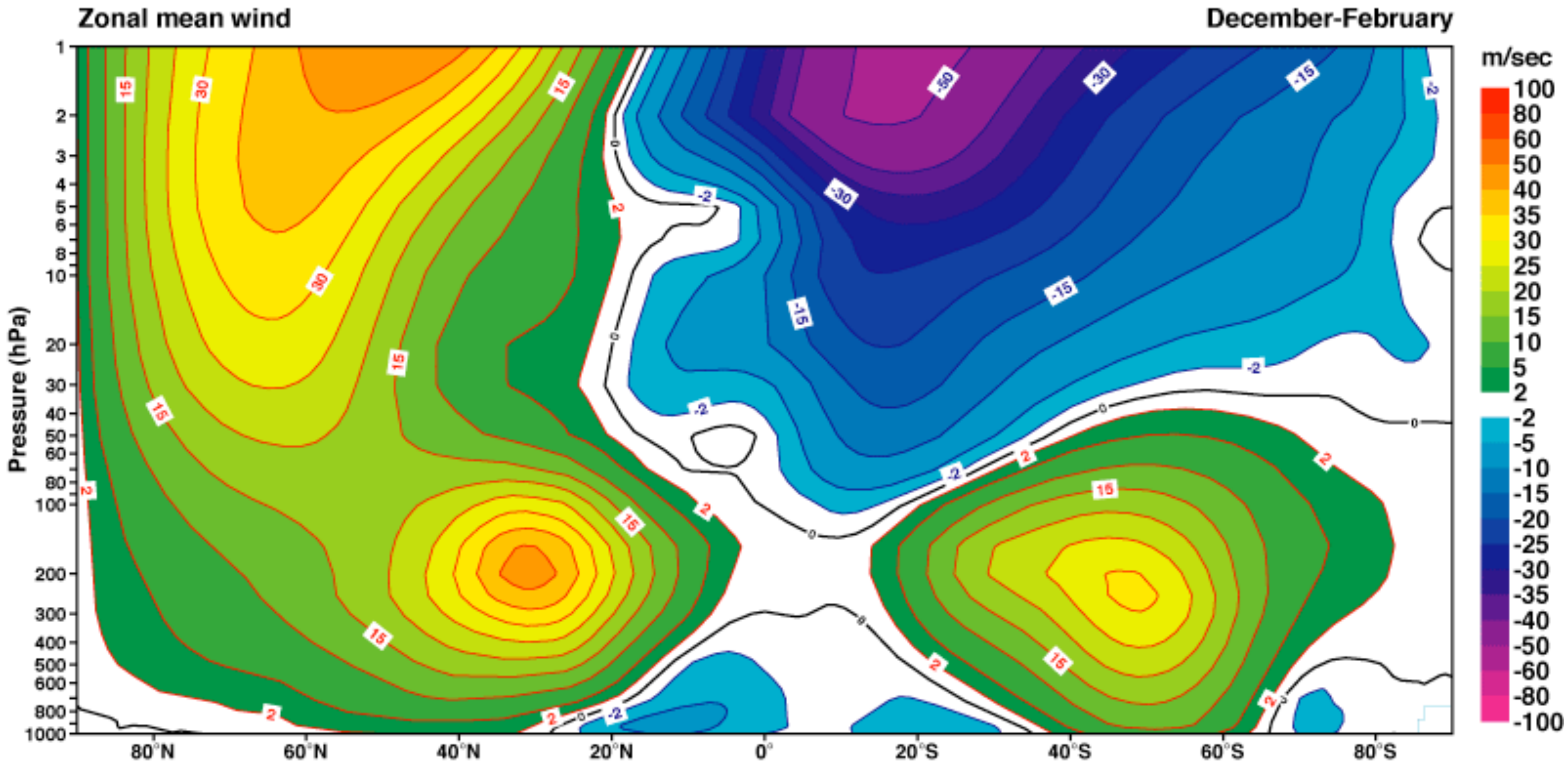


transient
eddies

$$[v^*T^*]$$



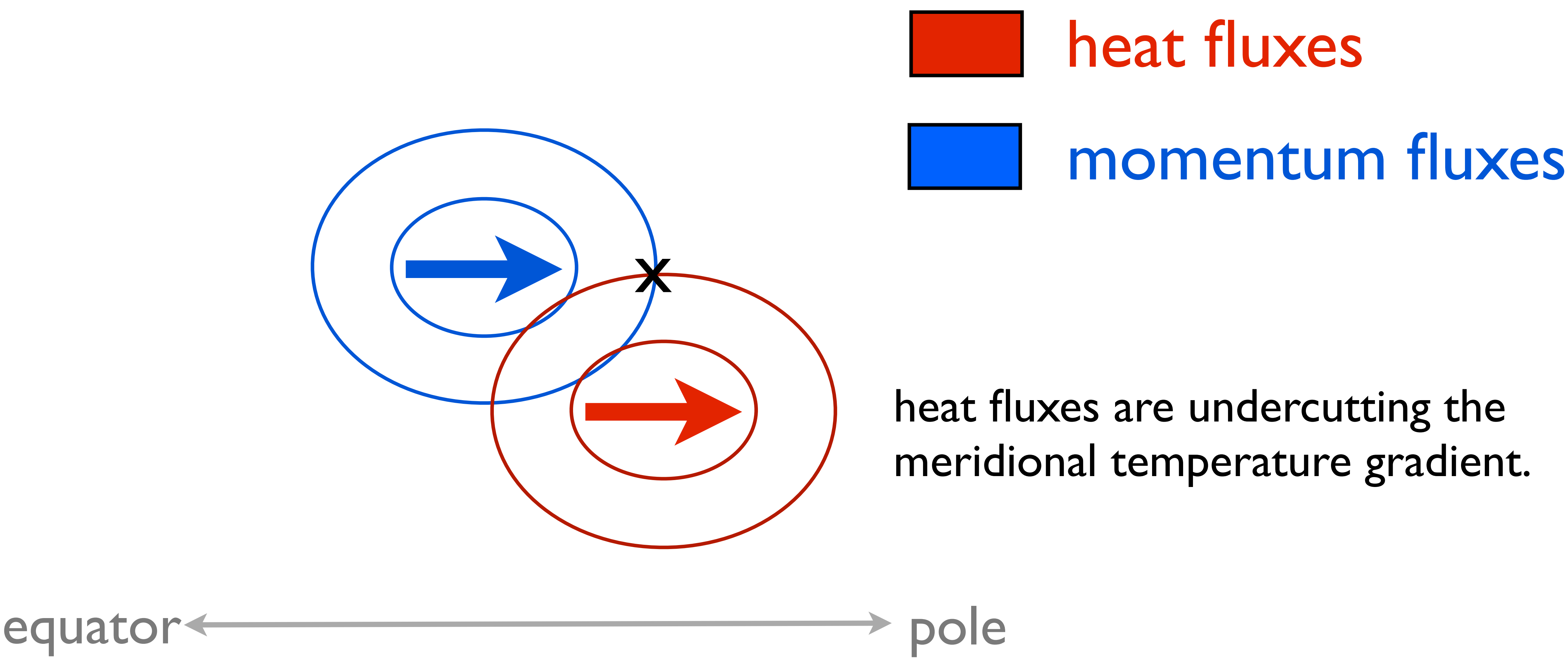
stationary
waves



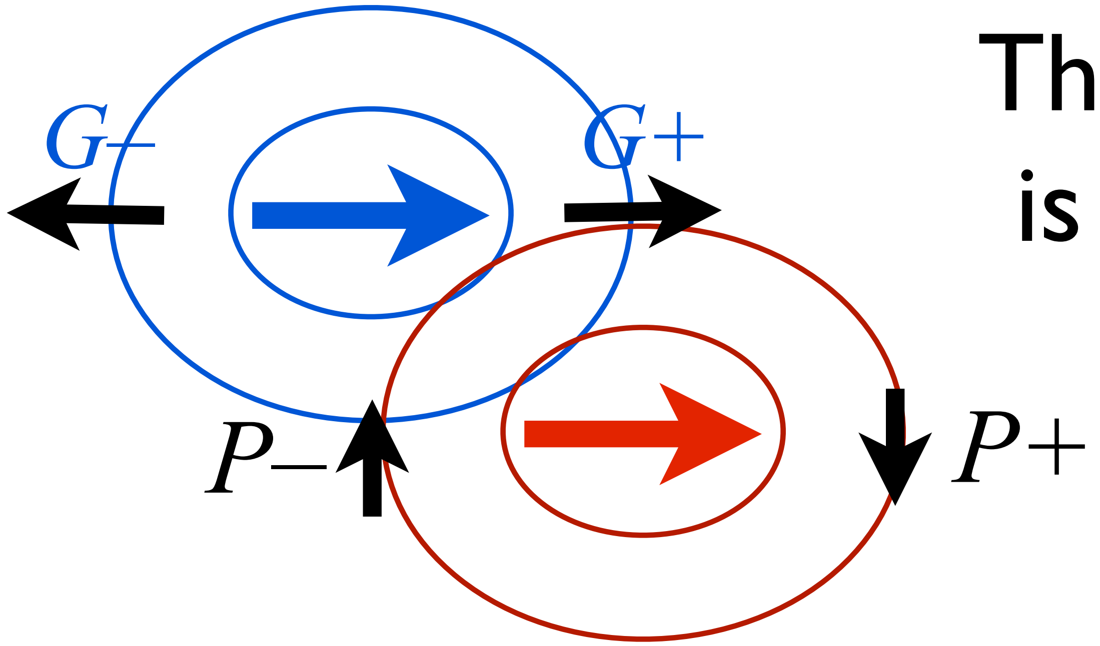
Zonal mean wind
December-February
m/sec
Pressure (hPa)
80°N
60°N
40°N
20°N
0°
20°S
40°S
60°S
80°S
100
80
60
50
40
35
30
25
20
15
10
5
2
-2
-5
-10
-15
-20
-25
-30
-35
-40
-50
-60
-80
-100

u

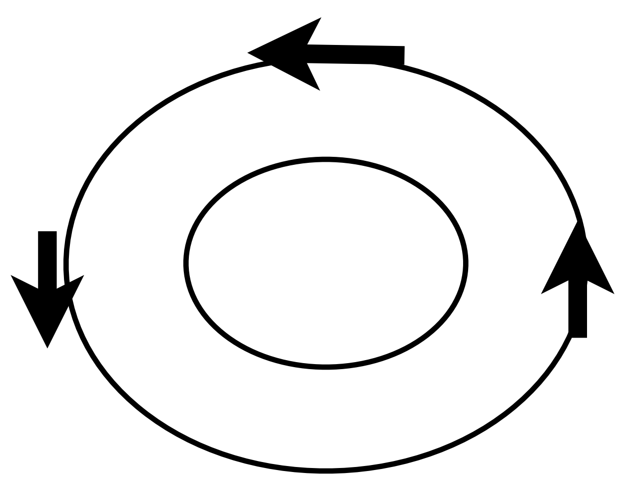
$$[q^* v^*] = -\frac{\partial}{\partial y}[u^* v^*] - f\frac{\partial}{\partial p}\frac{[v^* T^*]}{\sigma}$$



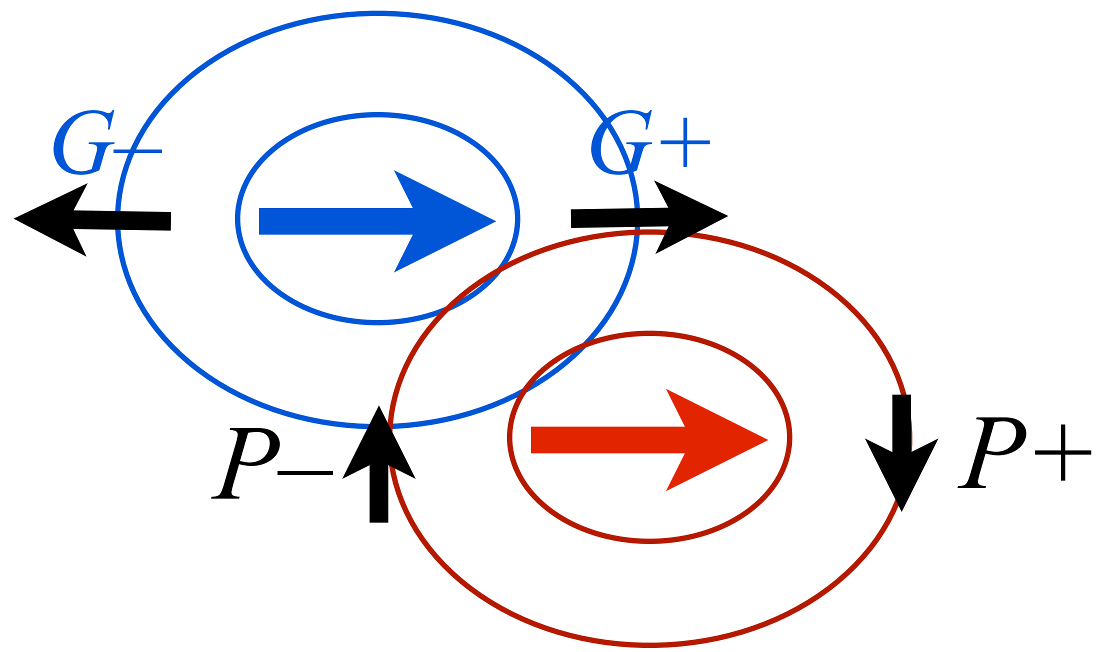
heat fluxes are undercutting the meridional temperature gradient.



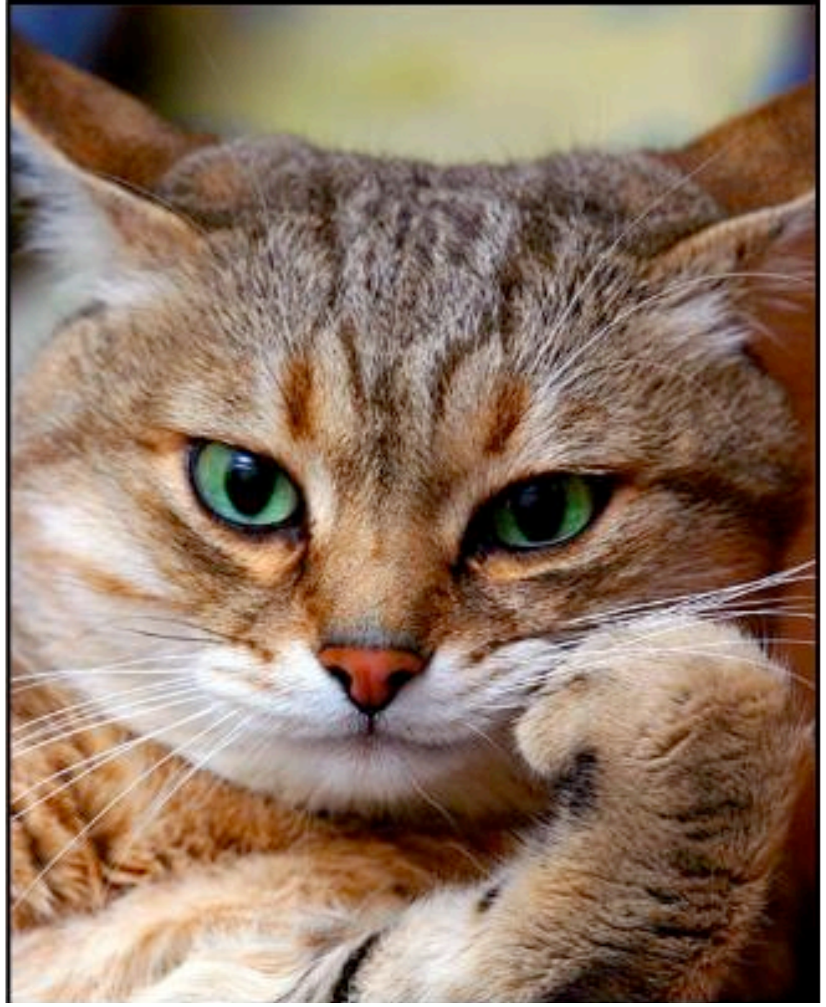
The eddy forcing vector is mainly nondivergent



and cancelled by the induced MMC

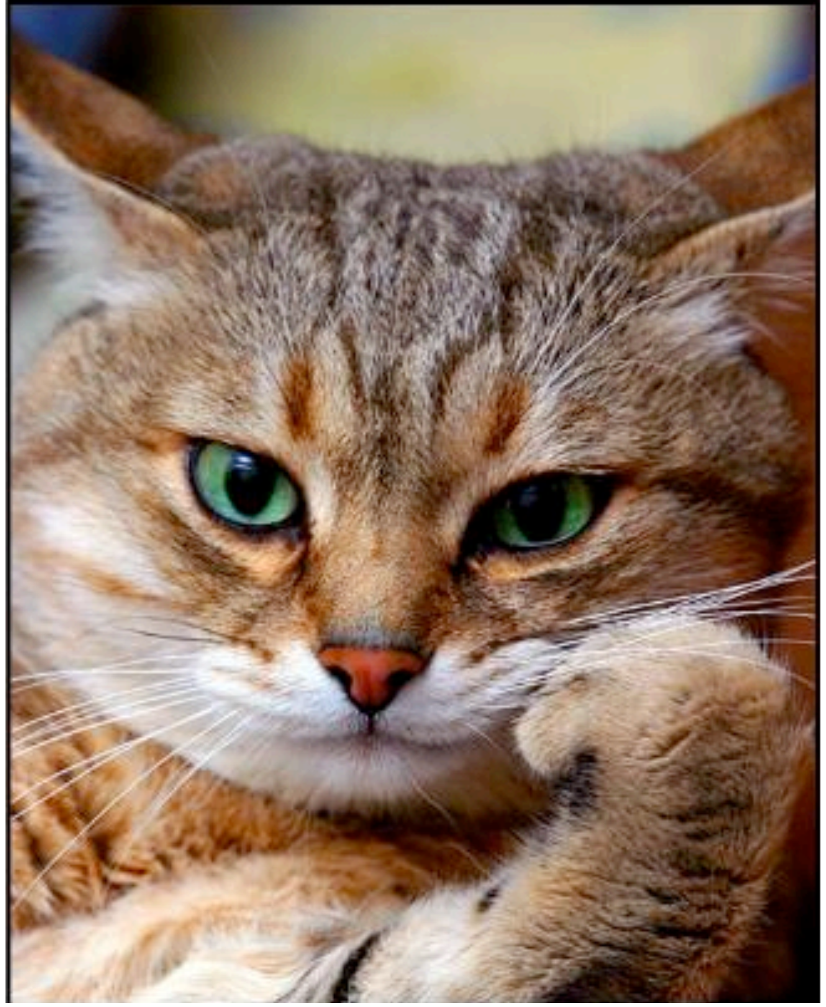


The momentum and heat fluxes are driving the flow out of thermal wind balance, increasing the vertical wind shear while weakening the meridional temperature gradient. The MMC won't let that happen.



Fluxes running wild and inducing MMC, yet nothing happens to the mean flow?

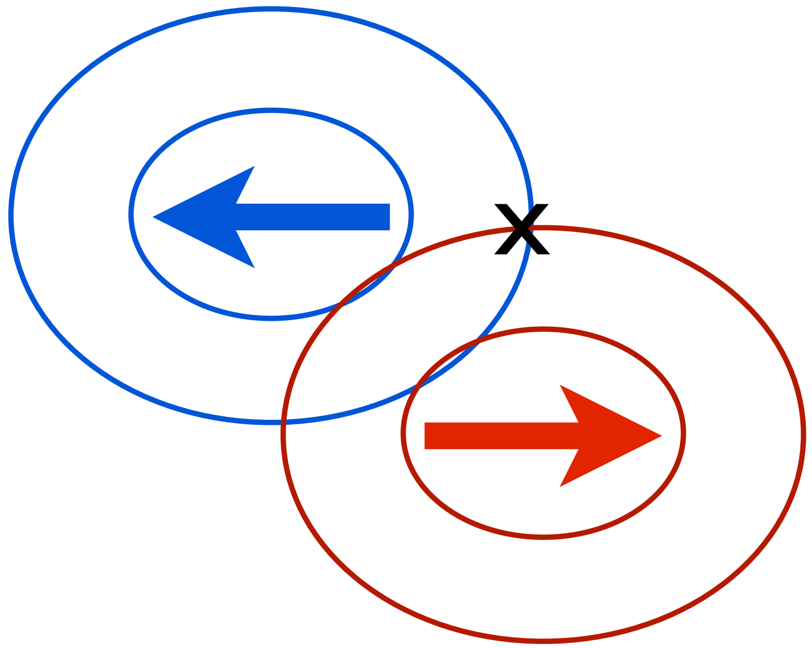
That’s right. But what if the momentum flux were equatorward rather than poleward?



All hell breaks loose?

Exactly!

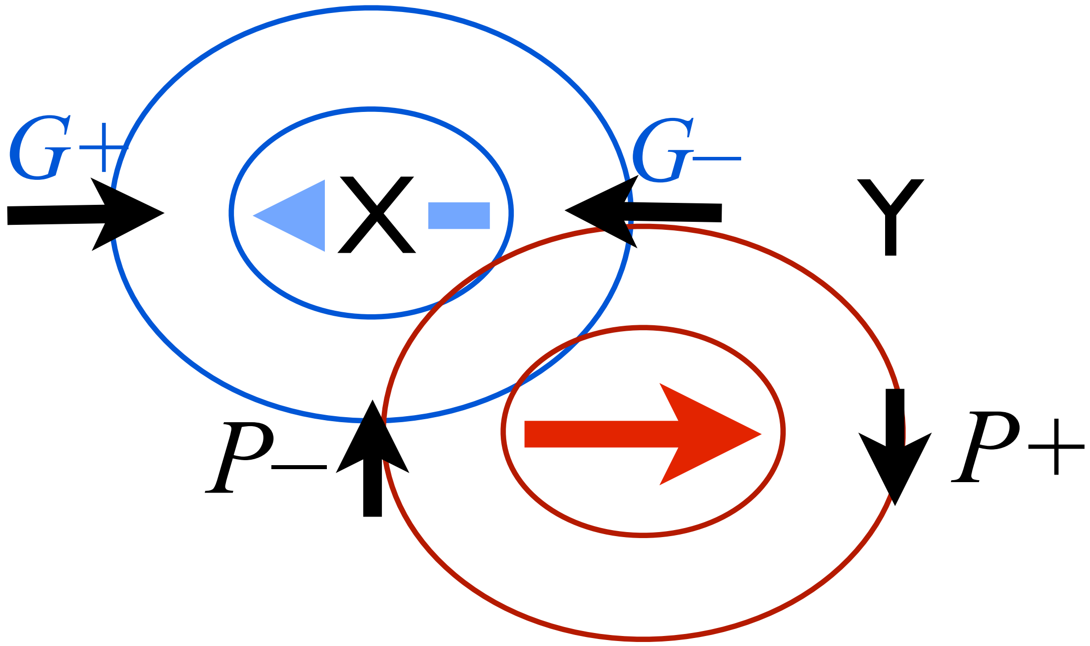
$$[q^* v^*] = -\frac{\partial}{\partial y}[u^* v^*] - f \frac{\partial}{\partial p} \frac{[v^* T^*]}{\sigma}$$



tendency for reenforcement

$[q^* v^*]$ is equatorward

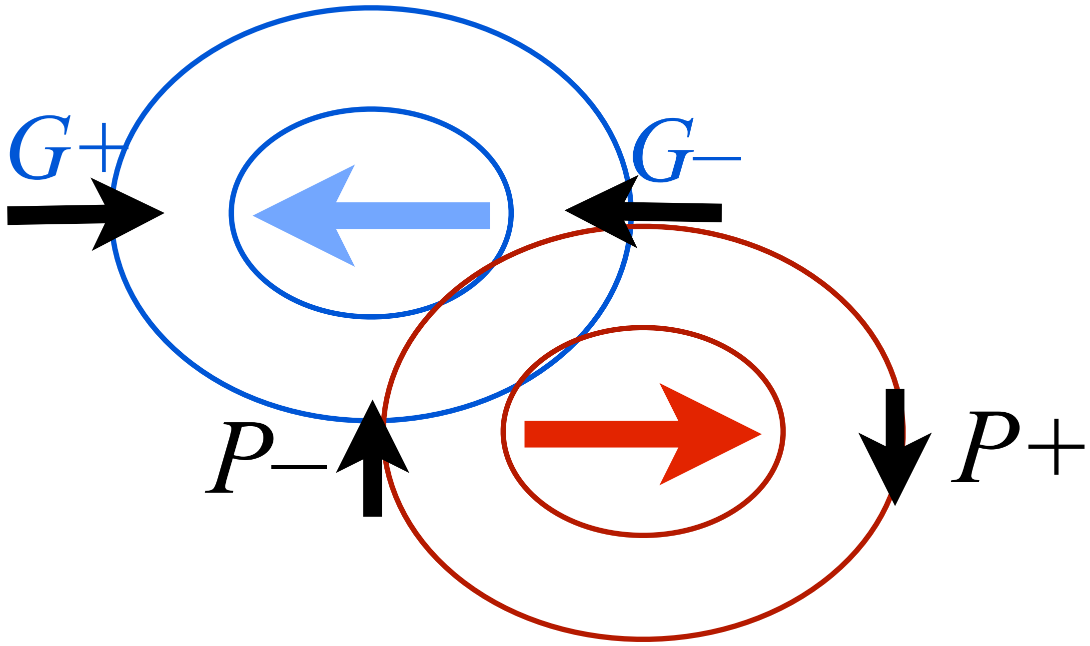
easterly acceleration



in this case the eddy forcing vector is mainly irrotational. The membrane thickens at X and thins at Y

The easterly acceleration occurs midway between X and Y, where the forcing is strongest.

in effect, membrane mass moves equatorward, consistent with $[q^* v^*] < 0$



in this case there is little or no eddy-induced MMC because the forcing field is nearly irrotational

The zonal flow isn't being forced away from thermal wind balance. The vertical shear and the meridional temperature gradient are both being forced to decrease.

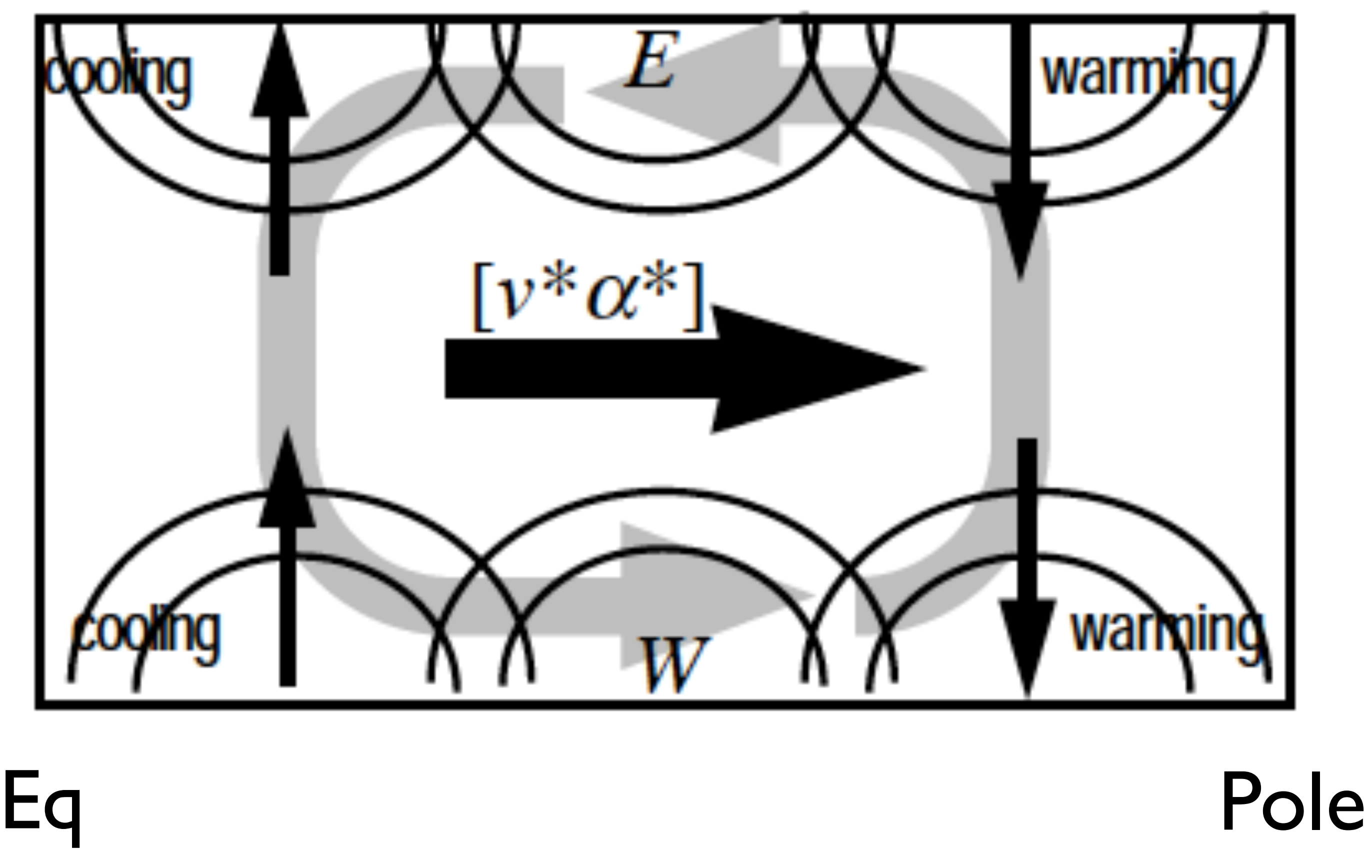
Do the momentum fluxes reverse in nature?

For week after week, the patterns resemble the climatology. The so-called *polar night jet* is even stronger than in the climatology (~50 m/s at 10 hPa) and temperatures over the polar cap region approach –80°C, the threshold for the formation of polar stratospheric clouds. The poleward eddy heat fluxes are strong, but the tendencies that they induce are almost exactly cancelled by the MMC.

Then every so often (on average once or twice per winter) the momentum fluxes reverse and within a few days, the polar vortex moves off the pole and weakens while temperatures over the polar cap region rise by 50°C or more.

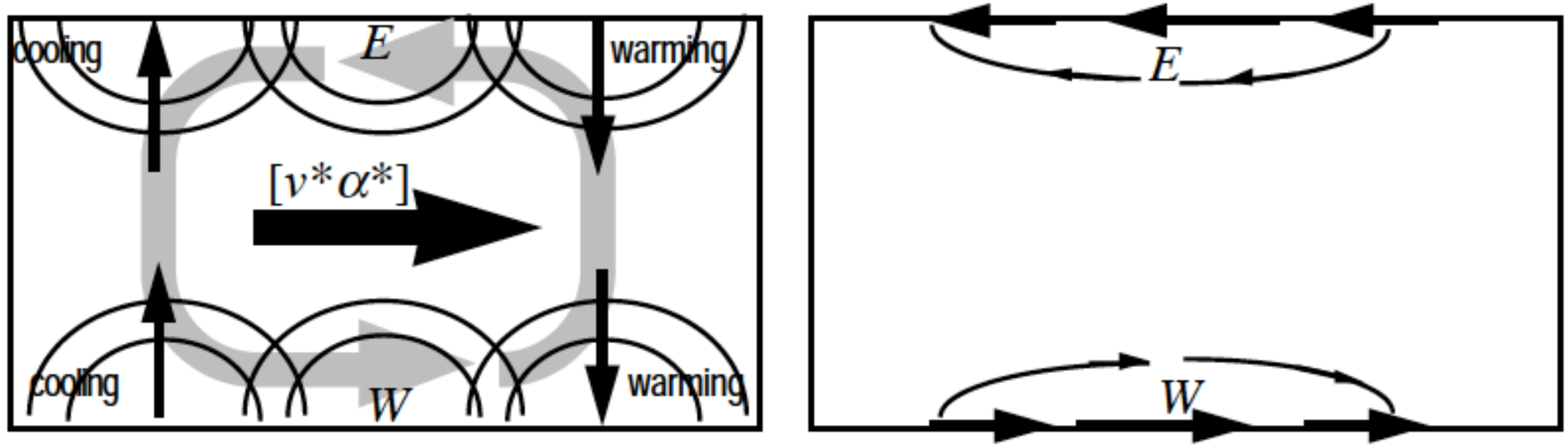
I'm intrigued with the bottom boundary condition. Can you tell us more about that?

It's those heat fluxes that we talked about earlier



Imagine the situation depicted in the left panel.
No momentum fluxes, friction, or diabatic heating.
Heat fluxes not varying with height.

The heat fluxes extend all the way to the top and bottom boundaries where the vertical component of the MMC has to vanish.

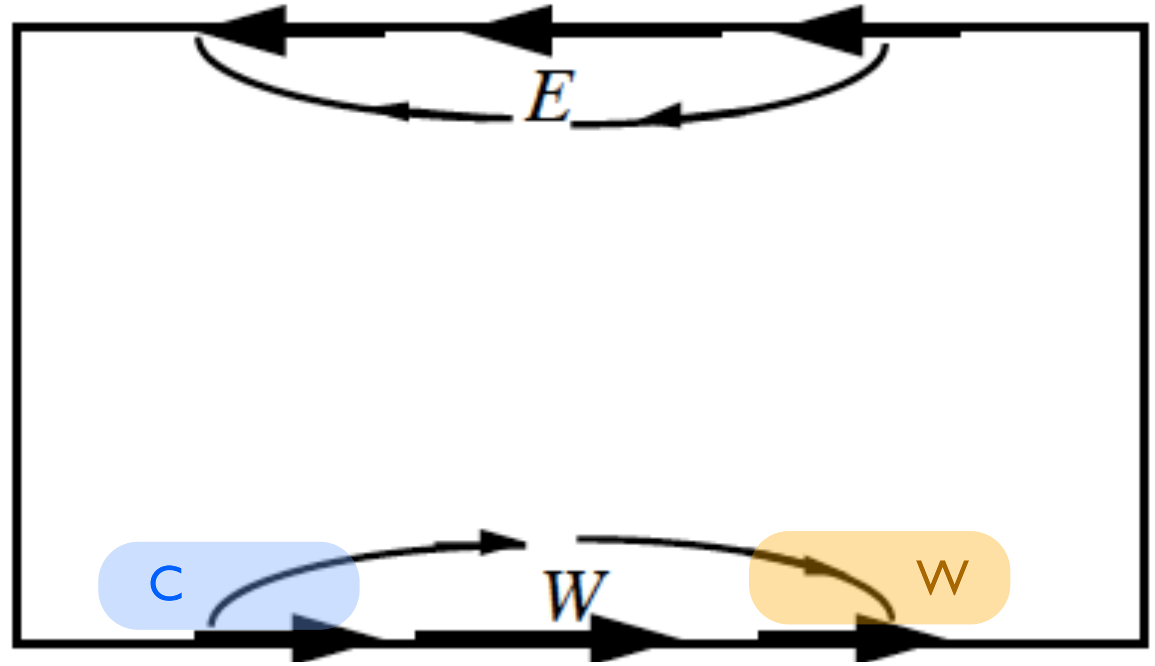


Eq Pole

To satisfy this constraint it is necessary to invoke a boundary-forced component of the solution.

We allow potential vorticity (membrane mass) to accumulate in reservoirs the top and boundaries as prescribed by the forcing vectors (left).

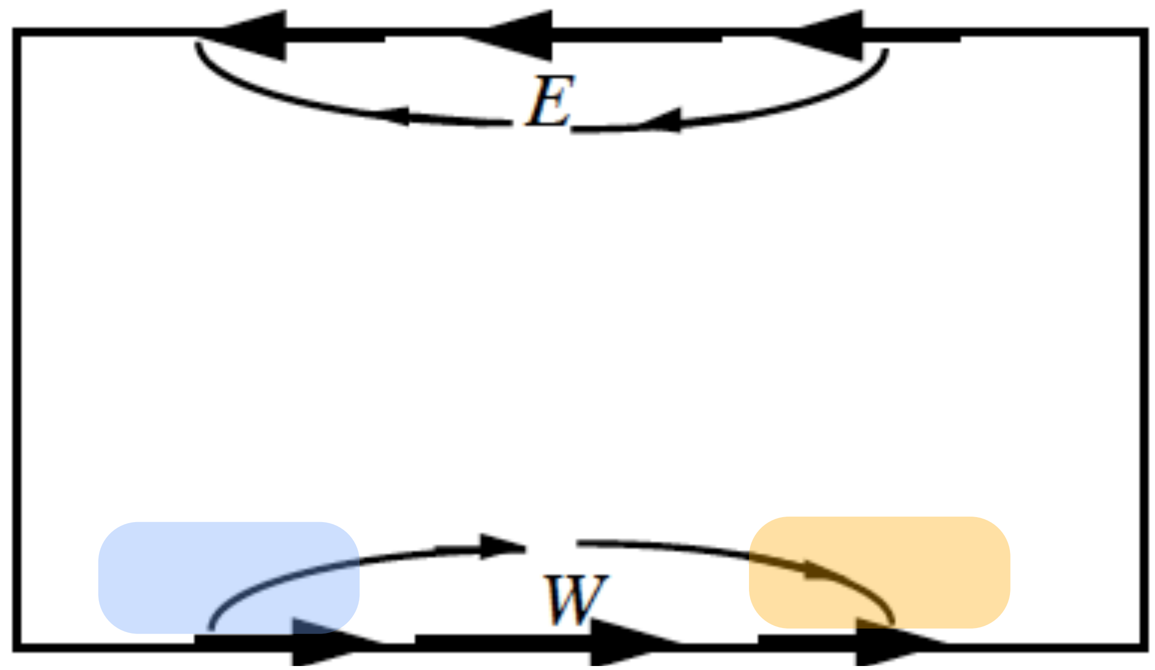
This is equivalent to moving membrane mass from left to right along the bottom boundary (right).



Because of the inherent stiffness of the membrane, mass must also move meridionally within some finite depth, as determined by the modulus of elasticity and there are associated vertical displacements of the membrane.

The poleward flux of potential vorticity induces a westerly acceleration on and just above the bottom boundary, as indicated.

The associated vertical velocities weaken the meridional temperature gradient just above the bottom boundary



Bearing in mind that the induced zonal wind and temperature tendencies are both weakening with distance from the top and bottom boundaries, the induced changes are consistent with geostrophy.

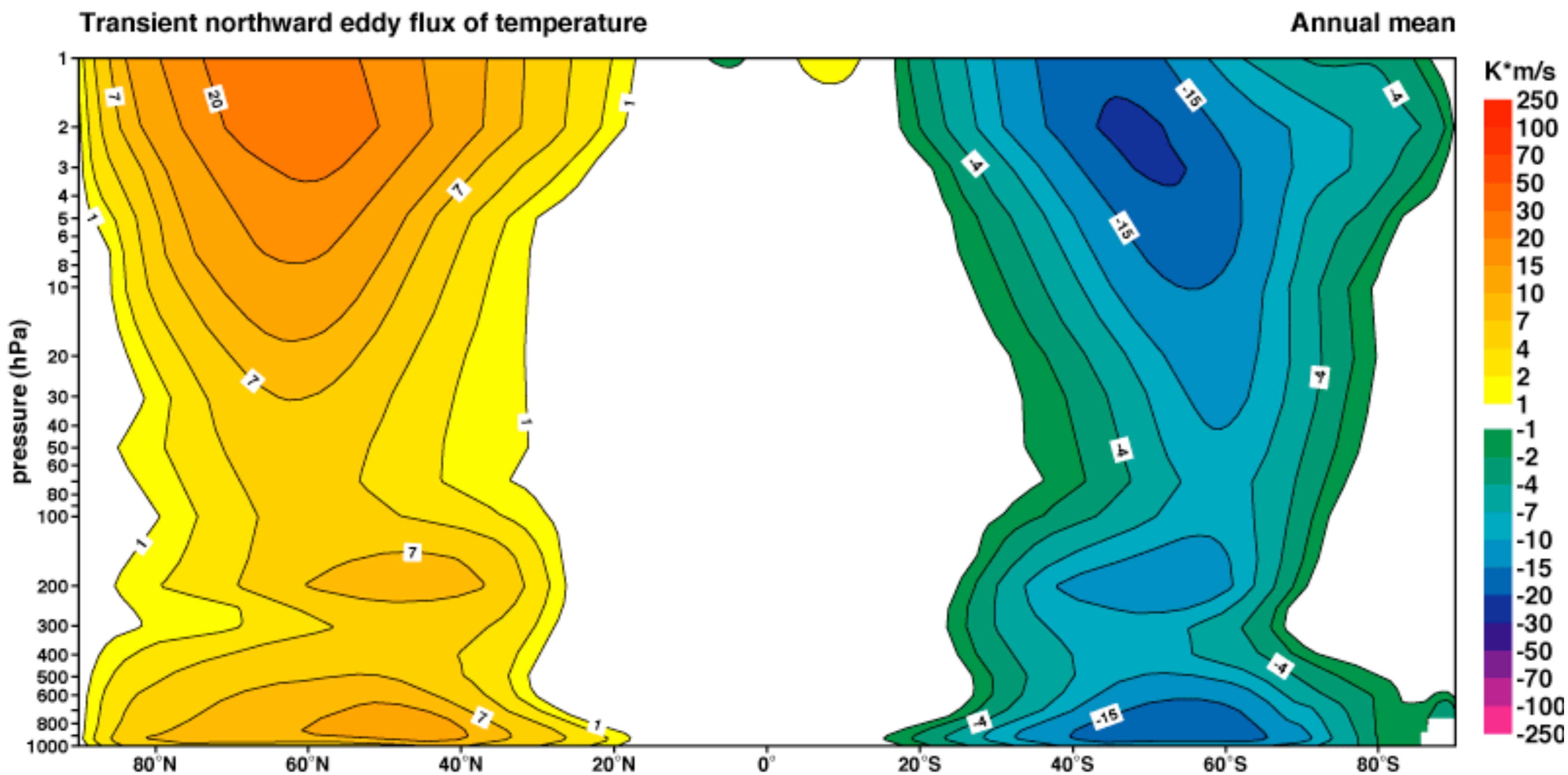
Above the bottom boundary, $\partial u / \partial z$ and $\partial T / \partial y$ are both weakened by the poleward heat flux on the boundary.

Smells like barotropification.

Exactly! The poleward heat flux at the bottom boundary, in combination with the heat fluxes in the interior and their induced MMC, weaken the meridional temperature gradient and vertical wind shear from the ground up.

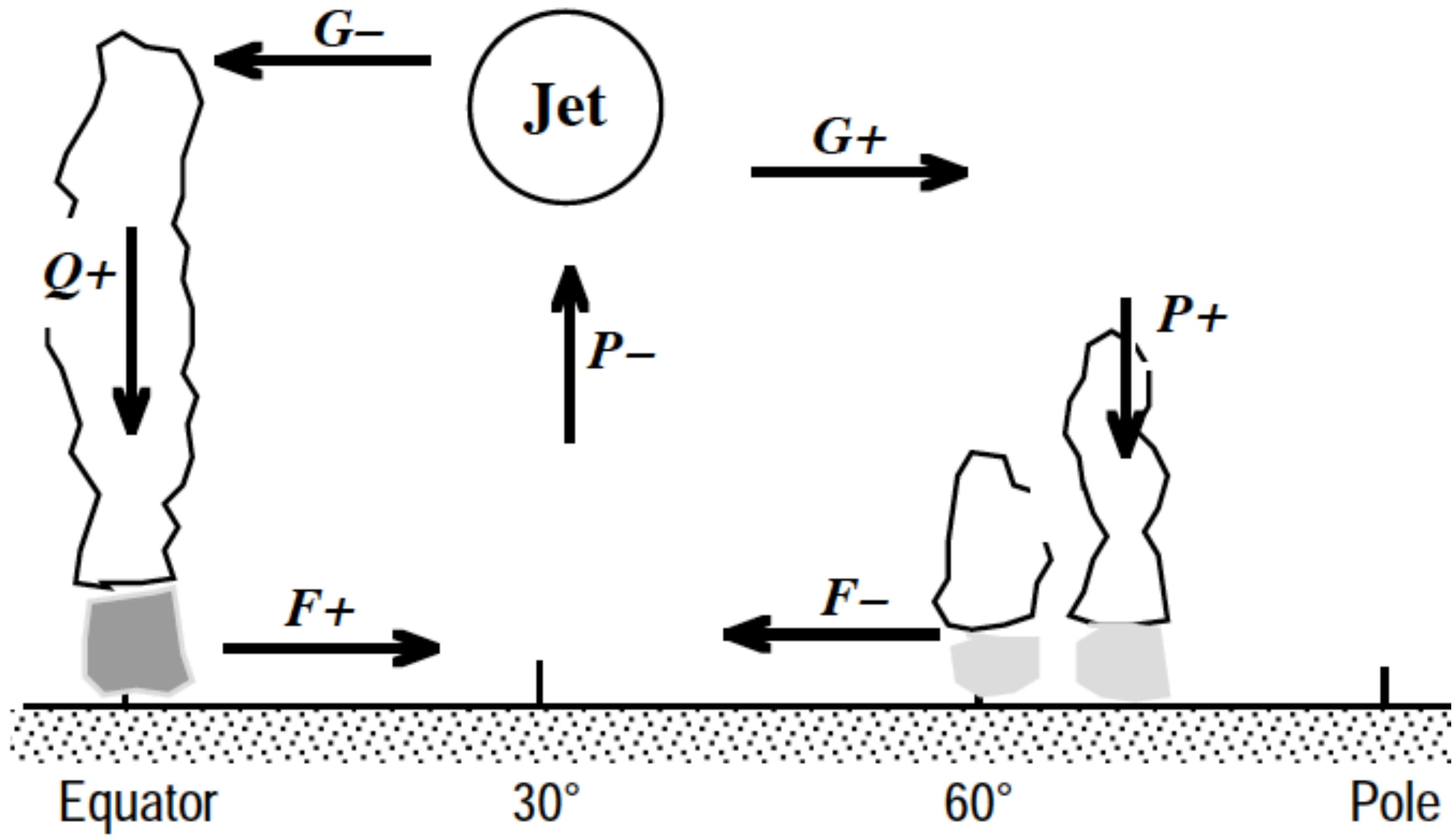
I think I’ve smelled this before.

You have. The geostrophic temperature advection at the bottom boundary plays an equivalent role in the quasi-geostrophic system and it is central to the theory of baroclinic instability.

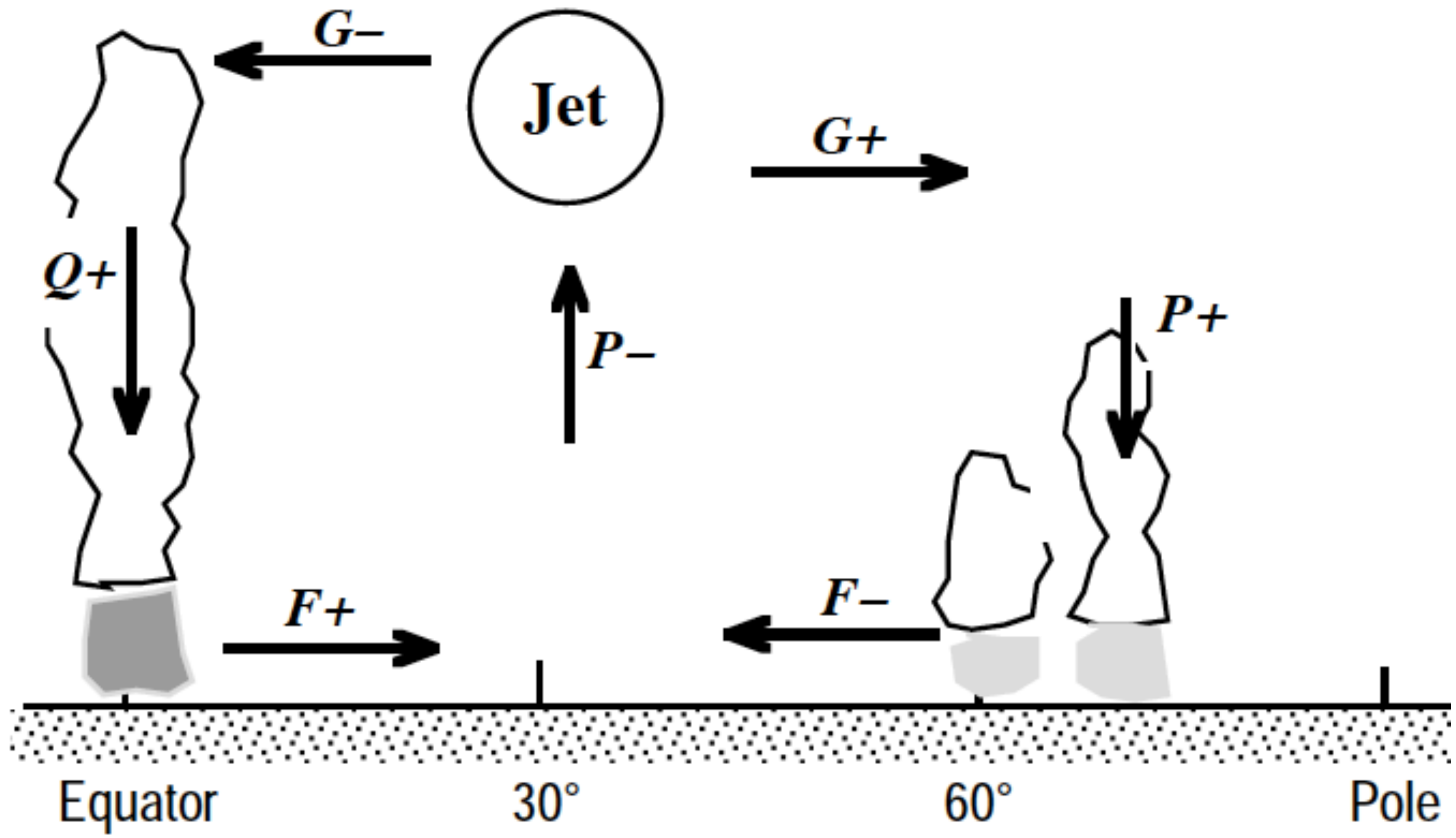


Note the maximum in the boundary layer.

Here are a few exercises to help it sink in.



Here's the climatology. Now what would happen if we suddenly double the strength of $[u^*v^*]$?



OK. Now let's do the same with $[v^*T^*]$? What happens?

The MMC always act to oppose the forcing.

Yet in the very act of opposing the forcing, they change the field that isn't being forced in the sense that is geostrophically consistent with the forcing.

In both the above examples, the system changes in response to the forcing but, thanks to the MMC, it does so in a geostrophically consistent way.

These conclusions apply equally well regardless of whether the forcing is due to eddy fluxes, diabatic heating, or friction.