APPENDIX 3

THE TOTAL ENERGY BALANCE EQUATION

The principle of the conservation of energy can be applied to an air parcel as it moves through the atmosphere. For such a parcel, we can write

$$\frac{dE}{dt} = Q - W \tag{A 3.1}$$

where E is the total energy content, which will be defined more precisely later on, Q is the rate of diabatic heating and W is the rate at which the parcel is doing work on its environment. All the above quantities are expressed per unit mass. For an infinitessimal block of air with dimensions δx, δy and δz, the rate at which work is being done by the zonal component of the motion is given by

$$\{pu|_{x + \delta x} - pu|_x\} \ \delta y \delta z$$

Expanding pu in a Taylor series expansion, for $\delta x \to 0$ the above expression can be written

$$\{\frac{\partial}{\partial x} (pu)\} \ \delta x \delta y \delta z$$

Treating the other components in a similar manner and dividing through by the mass of the block, we obtain

$$W = \frac{1}{\rho}(\frac{\partial}{\partial x} pu + \frac{\partial}{\partial y} pv + \frac{\partial}{\partial z} pw) \equiv \frac{1}{\rho} \nabla \cdot p\vec{c}$$

where $\vec{c}$ is the three dimensional velocity vector. Thus (A 3.1) can be rewritten as

$$\rho \frac{dE}{dt} = \rho Q - \nabla \cdot p\vec{c} \tag{A 3.2}$$

which is the basis for the derivation of the First Law of Thermodynamics (See Fleagle and Businger p. 37). Making use of the identity

$$\rho \frac{d(\ )}{dt} = \frac{\partial}{\partial t} \rho(\ ) + \nabla \cdot \rho(\ ) \vec{c}$$

(A 3.2 can be rewritten in the form

$$\frac{\partial}{\partial t} \rho E = -\nabla \cdot (\rho E + p) \vec{c} + \rho Q \qquad \text{(A 3.3)}$$

where

$$E \equiv c_v T + \Phi + \frac{c^2}{2} \qquad \text{(A 3.4)}$$

There is a certain amount of arbitrariness in our definition of E. We could, for example, have included chemical potential as part of E. However, throughout the lower atmosphere there is very little conversion between chemical potential energy and the types of energy included in (A 3.4). The same is true of electrostatic and magnetic potential energies. If we had wished, we might have chosen to exclude gravitational potential energy Φ as part of E in (A 3.4) and instead, included the work term $-\rho\vec{c} \cdot \nabla\Phi$ in (A 3.2). It is readily verified that the balance statement would then have assumed the form

$$\frac{\partial}{\partial t} \rho \hat{E} = -\nabla \cdot (\rho \hat{E} + p + \rho\Phi) \vec{c} + \rho Q$$

where

$$\hat{E} \equiv c_v T + \frac{c^2}{2}$$

Note that the above formulation is essentially equivalent to (A 3.3-4) provided that $\partial\Phi/\partial t = 0$.

Now if we make use of the identities $p = \rho RT$ and $c_v + R = c_p$, we can write (A 3.3) in the form

$$\frac{\partial}{\partial t} \rho E = -\nabla \cdot \rho E' \vec{c} + \rho Q \qquad \text{(A 3.5)}$$

where

$$E' \equiv c_p T + \Phi + \frac{c^2}{2} \qquad \text{(A 3.6)}$$

Thus, in the absence of heat sources and sinks, the time rate of change of total energy E is governed by the advection of E'. Let us examine this quantity in more detail.

Recall, that the speed of sound is given by

$$c_s = \frac{c_p}{c_v} RT = \frac{R}{c_v} (c_p T)$$

where $R/c_v \simeq 2/5$. Thus (A 3.6) can be written as

$$E' = c_p T \; [1 + 0.2 \; (\frac{c}{c_s})^2] + \Phi$$

In the earth's atmosphere, where c is on the order of tens of m s^{-1}, the kinetic energy term is usually less than a percent of the enthalpy term. Hence, to a fairly close approximation†, $E' \simeq c_p T + \Phi$.

The quantity $(c_p T + \Phi)$ is called static energy. It is clear from the above discussion that it isn't really a form of energy, but, rather, it is a hybrid expression involving internal and gravitational potential energy plus work. Static energy has two properties which endear it to the heart of the atmospheric thermodynamicist:

- Under hydrostatic conditions, in the absence of diabatic heating it is a quasi conservative quantity. This property is evident from noting that since $\partial p/\partial t << dp/dt$, $\partial/\partial t(\rho E) \simeq \partial/\partial t(\rho E')$ and therefore (A 3.5) is approximately equivalent to a conservation equation for E'. Furthermore, it is readily shown that to a close approximation

†In strong jet streams, where c/c_s ranges up to about 0.2, air parcels experience time rates of change of kinetic energy that are within an order of magnitude of the time rates of change of the other terms in (A 3.5). Hence in careful analyses of the energetics of jet-streams, the kinetic energy term cannot be neglected.

$$E' \simeq c_p\theta \tag{A 3.7}$$

- In an unsaturated environment the vertical gradient of static energy is a measure of the static stability.

$$\frac{\partial}{\partial z}(c_pT + \Phi) = c_p(\frac{\partial T}{\partial z} + \frac{g}{c_p}) = c_p(\Gamma_d - \Gamma) \tag{A 3.8}$$

The transport of water vapor in the atmospheric branch of the hydrologic cycle plays an important role in the global energy balance. This transport is not explicitly taken into account in the advection term of (A 3.5-6). We can remedy that deficiency by making a few minor changes in the foregoing derivation. Let us begin by rewriting (A 3.1) in the form

$$\frac{d}{dt}(c_vT + \Phi + c^2/2) = (Q_R + S_h + L.H.) - W \tag{A 3.9}$$

Where Q_R is the net radiative heating rate, S_h is the heat gained or lost by conduction from or to the underlying surface and L.H. is the latent heat of condensation derived from precipitation processes taking place within the atmosphere. Now we write

$$L.H. = -L(\frac{dq}{dt})_A = -L[\frac{dq}{dt} - (\frac{dq}{dt})_G] \tag{A 3.10}$$

where q is the specific humidity of the air and the subscripts A and G refer to phase changes taking place within the atmosphere, and at the earth's surface, respectively. Substituting back into (A 3.9), and rearranging terms slightly, we obtain

$$\frac{d}{dt}(c_vT + \Phi + \frac{c^2}{2} + Lq) = (Q_R + S_m + S_h) - W \tag{A 3.11}$$

where $S_m \equiv L(\frac{dq}{dt})_G$ represents the latent heat flux from the ground to the atmosphere.

With (A 3.11) as a starting point we could repeat the derivation of (A 3.5), which now takes the form

$$\frac{\partial}{\partial t}\rho\,(c_v T + Lq + \Phi + \frac{c^2}{2}) = -\nabla\cdot\rho(c_p T + qL + \Phi + \frac{c^2}{2})\,\vec{c} + \rho(Q_R + S_m + S_h) \qquad \text{(A 3.12)}$$

If we ignore the small kinetic energy term, the quantity that is advected is seen to be the sum of the static energy $(c_p T + \Phi)$ plus the latent heat content Lq. This term appears so frequently in atmospheric thermodynamics that it is given the name <u>moist static energy</u>.

Let us consider the physical interpretation of moist static energy. Since $\partial p/\partial t << dp/dt$ in the atmosphere, it follows that $\partial/\partial t(\rho c_v T) \simeq \partial/\partial t(\rho c_p T)$ and therefore (A 3.12) is approximately equivalent to the statement that

$$\frac{\partial}{\partial t}\,\rho E'' = -\nabla\cdot\rho E''\vec{c} + \rho(Q_R + S_m + S_h) \qquad \text{(A 3.13)}$$

where E'' is moist static energy. Or again making use of the identity prior to (A 3.3)

$$\frac{dE''}{dt} \approxeq Q_R + S_m + S_h \qquad \text{(A 3.14)}$$

We can interpret this result as follows: When condensation or evaporation takes place within the atmosphere there is a conversion between the sensible heat $c_p T$ and the latent heat content Lq, but the moist static energy remains constant. Moist static energy can be changed only by radiative heating or by the transfer of heat or moisture from the underlying surface.

It is interesting to note that $E'' \simeq c_p\theta_e$ (the identity is exact if the 1000 mb surface coincides with sea level, and if the small variation of L with temperature is ignored). We recall that $\partial\theta_e/\partial z < 0$ is the criterion for convective instability. It follows that, in a convectively stable atmosphere, moist static energy increases with height.

Now let us return to the exact equation (A 3.12) and integrate it over volume, using the hydrostatic equation to convert to pressure as a vertical coordinate:

$$\frac{d}{dt}\iiint (c_vT + Lq + \Phi + c^2/2)\,dxdydp = \iint \left(c_pT + Lq + \Phi + \frac{c^2}{2}\right) c_{in}ds + \iint (Q_R + S_m + S_h)\,dxdydp \qquad \text{(A 3.15)}$$