Atmospheric Science 582—Winter 2014

Homework 3 – (Take home midterm, work independently)

1. Consider piecewise linear polynomial reconstructions in which the reconstructed solution within each finite volume $x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ is given by

$$\tilde{\phi}(x) = \phi_j^n + \sigma_j^n (x - x_j),$$

where ϕ_j and σ_j are the mean and slope over the volume. Let the slope be defined as

$$\sigma_j = \frac{\phi_{j+1} - \phi_j}{\Delta x}\, C(r_{j+\frac{1}{2}}),$$

where $C(r)$ is the limiter and for positive wind speeds,

$$r_{j+\frac{1}{2}} = \frac{\phi_j - \phi_{j-1}}{\phi_{j+1} - \phi_j}.$$

Forward and backward facing slopes will be treated the same if, $\forall r > 0$, the limiter satisfies

$$C\left(\frac{1}{r}\right) = \frac{C(r)}{r}. \tag{1}$$

Prove that this statement is true. *Hint:* The goal is to show that (1) implies $\sigma_j = -\sigma_k$ whenever $\phi_{j+s} = \phi_{k-s}$ for all s in the stencil used to determine the slopes.

2. The Matlab routines `starting_point.m` and `Lax_Wendroff.m` compute solutions to the constant windspeed advection equation in flux form

$$\frac{\partial \psi}{\partial t} + \frac{\partial c\psi}{\partial x} = 0,$$

on the periodic domain $x \in [-1, 1]$ for the case $c = 1$. As in class, the slope limiter approach is written in an equivalent flux form such that

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0,$$

where

$$F_{j+\frac{1}{2}}^n = c\phi_j^n + \frac{c}{2}(1-\mu)\left(\phi_{j+1}^n - \phi_j^n\right) C(r_{j+\frac{1}{2}}^n),$$

and $\mu = c\Delta t/\Delta x$ is the Courant number. Two alternative initial conditions are considered

$$\psi(x, 0) = \left(\frac{1 + \cos(\pi x)}{2}\right)^4$$

and the square-wave periodic continuation of

$$\psi(x,0) = \frac{1 + H\left(\cos(\pi x)\right)}{2},$$

where H is the Heaviside step function.

The routines use $\Delta x = 0.04$ and a Courant number of 0.5 to integrate to time $t = 4$. The code provided assumes $C(r) = 1$.

(a) Show solutions at time $t = 4$ for both initial conditions, and for the exact, the unlimited Lax Wendroff solution, and limited solutions where

$$C(r) = \max\left(0, \min(r, \beta)\right)$$

with $\beta = 1$, 1.5 and 2.

(a) Compare the preceding to solutions obtained using the limiter

$$C(r) = \max\left(0, \min(r, \beta), \min(\beta r, 1)\right)$$

where again $\beta = 1$, 1.5 and 2.

(c) Why do some of the preceding flux-limiter (or equivalently slope-limiter) schemes avoid phase errors while the underlying unlimited scheme, and other limiter formulations, do not?

Due Friday, February 20