Atms 536: Homework 1: Deformation and the Axis of Dilatation

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1. Consider a horizontal flow field with x- and y-component velocities (u, v). The stretching and the shearing deformation are defined respectively as

$$D1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \qquad D2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

Consider a counterclockwise rotation of the coordinate axes through an angle ϕ such that the new coordinates become

$$x' = x\cos\phi + y\sin\phi, \qquad y' = y\cos\phi - x\sin\phi.$$

Let (u', v') be the velocities along the rotated coordinates and define

$$D1' = \frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial y'}, \qquad D2' = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}.$$

(a) Derive expressions for $D1$ and $D2$ in terms of $D1'$, $D2'$ and ϕ.

(b) Using the result from (a), derive expressions for $D1'$ and $D2'$ in terms of $D1$, $D2$ and ϕ.

(c) How do the values of the stretching and shearing deformation change when the coordinates are rotated 45°counterclockwise?

(d) Show that the total deformation

$$D = \left(D1^2 + D2^2\right)^{1/2}$$

is invariant under coordinate rotation, and hence more physically meaningful than $D1$ or $D2$ themselves.

(e) Assuming $D1 > 0$, show that the axis of dilatation along which the total deformation stretches a scalar field, is located at an angle α measured counterclockwise from the x-axis and satisfying

$$\alpha = \frac{1}{2}\arctan(D2/D1). \tag{1}$$

Hint: rotate the axes through an angle that makes $D2' = 0$.

Note that in the case $D1 < 0$, this same rotation will yield the axis of contraction, along which the total deformation increases a scalar gradient most rapidly.

2) Explore the orientation of the axis of dilatation associated with different flow fields. Use the Matlab file linked on the website to get started.

The Matlab file has four stream-function fields coded into it: a "localized straight jet," a "square wave with deformation", a "sinusoidal wave with uniform wind speeds", and a more complex "realistic case." The Matlab code computes velocities from the stream function ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

and plots velocity vectors plots together with contours of the stream function.

Your assignment is to code up the computation of the total deformation and to plot it as a vector (actually a line segment without arrowheads) aligned with the axis of dilatation for each point at which the wind vectors are plotted in the sample plot. Superimpose this on contours of the stream function. A snippet of code is commented out at the end of the program that can do this plotting for you. Submit one plot for each of the four stream-function patterns along with your Matlab code. Comment briefly on the types of deformation associated with the various flow patterns.

To correctly distinguish between axes of dilatation and contraction you need to worry about the signs of $D1$ and $D2$. One way to deal with this is to consider the difference between the matlab commands `atan` and `atan2`.

How to figure out if your code is working correctly: Start with the simplest cases, the localized straight jet and the square wave with deformation. Contour plot the individual components D1 and D2 for these cases and be sure your axes of dilatation are correct in these simple cases.

Due Thursday, February 2nd.