Atms 536: Homework 2

---

1. *Virtual temperature – with liquid water.* The total density ρ is the sum of the densities of dry air ρ_d, water vapor ρ_v, and liquid water ρ_l. Let $r_v = \rho_v/\rho_d$ and $r_l = \rho_l/\rho_d$ be the mixing ratios of water vapor and liquid water respectively, and define $\epsilon = R_d/R_v$ as the ratio of the gas constants for dry air and water vapor.

Show that if the virtual temperature

$$T_v = T\left(\frac{1 + r_v/\epsilon}{1 + r_v + r_l}\right)$$

is used in the equation of state

$$p = \rho R_d T_v$$

one obtains the correct total pressure (which is the sum of the partial pressures exerted by the dry air and the water vapor) for an air parcel containing both water vapor and liquid droplets.

2. *Expressions for Buoyancy.* The virtual potential temperature of an air parcel is defined

$$\theta_v = T_v(p/p_0)^{-R/c_p},$$

where T_v is the virtual temperature, the gas constant for dry air is denoted by R for simplicity, c_p the specific heat of air at constant pressure and p_0 a constant reference pressure typically specified as 10^5 Pascals (1000 mb).

(a) The Exner function pressure is defined

$$\pi = (p/p_0)^{R/c_p}.$$

Show that

$$c_p\theta_v\nabla\pi = \frac{1}{\rho}\nabla p.$$

Note, for use in (b), that this implies the momentum equations for inviscid airflow may be written in the form

$$\frac{D\mathbf{v}}{Dt} + c_p\theta_v\nabla\pi = -g\mathbf{k}.$$

(b) Show that after removing a hydrostatically balanced reference state $\overline{\theta}_v(z)$ and $\overline{\pi}(z)$ the momentum equations for inviscid flow may be written

$$\frac{D\mathbf{v}}{Dt} + c_p\theta_v\nabla\pi' = g\frac{\theta_v'}{\overline{\theta}_v}\mathbf{k}. \quad (1)$$

(The advantage of this form is that it links virtual potential temperature perturbations directly to buoyancy perturbations without *any* approximations to the governing equations.)

In (c) and (d) below, focus on the dry case, for which $\theta = \theta_v$.

(c) Except in the expression for buoyancy itself, one can often assume that the vertically varying reference state thermodynamic variables are much larger that the perturbations, i.e.,

$$\theta' \ll \overline{\theta}, \quad \rho' \ll \overline{\rho}, \quad p' \ll \overline{p}$$

etc. Under this approximation (1) becomes

$$\frac{D\mathbf{v}}{Dt} + c_p \overline{\theta} \nabla \pi' = g \frac{\theta'}{\overline{\theta}} \mathbf{k}. \tag{2}$$

Similarly, the vertical momentum equation expressed in terms of ρ and p becomes

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\overline{\rho}} \nabla p' = -g \frac{\rho'}{\overline{\rho}} \mathbf{k}. \tag{3}$$

Show that the expressions for the buoyancy on the right sides of (2) and (3) are not equivalent, i.e., that (2) and (3) split the total forcing for the acceleration between the vertical pressure gradient and buoyancy in different ways.

(d) Nevertheless, in traditional arguments about the buoyancy in rising air parcels, the buoyancy expressions in (2) and (3) become identical. Why is this?

3. *Water loading.* There are two different ways to approach the vertical momentum equation when hydrometeors are present. One approach is to write the momentum equation for gaseous matter alone and to explicitly include the frictional drag exerted on the gas by falling precipitation. The second approach is to generalize the expression for virtual temperature to include the non-gaseous components and to compute buoyancy perturbations in using generalized virtual temperatures. Below we examine the question: *to what extent are these two approaches equivalent?*

Let over-bars again denote a reference state that varies only in z and suppose that there is no liquid water in the reference state (only vapor). As notation for this problem, let $\theta_{\tilde{v}}$ approximate the virtual temperature expression including liquid water

$$\theta_{\tilde{v}} = \theta_d \left(1 + 0.61 r_v - r_l\right)$$

and let θ_v approximate the virtual temperature without accounting for liquid water

$$\theta_v = \theta_d \left(1 + 0.61 r_v\right).$$

The vertical momentum equation treating the acceleration of the gases and the liquids together via virtual temperature may be written as

$$\frac{Dw}{Dt} + c_p \overline{\theta}_{\tilde{v}} \frac{\partial \pi'}{\partial z} = g \left(\frac{\theta_{\tilde{v}} - \overline{\theta}_{\tilde{v}}}{\overline{\theta}_{\tilde{v}}} \right) \tag{4}$$

The vertical momentum equation treating the acceleration of the gases alone with a drag term due to the falling liquid droplets may be written as

$$(\rho_d + \rho_v) \left[\frac{Dw}{Dt} + c_p \overline{\theta}_v \frac{\partial \pi'}{\partial z} - g \left(\frac{\theta_v - \overline{\theta}_v}{\overline{\theta}_v} \right) \right] = F, \tag{5}$$

where F is the frictional drag per unit volume exerted by the falling droplets.

(a) The full virtual temperatures in the pressure-gradient terms in (4) and (5) have been approximated by their reference-state values (as they sometimes are in numerical models). What is the relation between $\overline{\theta}_v$ and $\overline{\theta}_{\tilde{v}}$?

(b) Assuming that all droplets are falling at their terminal velocity, derive an expression for F.

(c) Use your expression for F to show that the buoyancy term in (4) and (5) may be expressed as

$$g \left[\frac{\theta_d(1 + .61 r_v - \mathcal{H} r_\ell - \overline{\theta}_v}{\overline{\theta}_v} \right],$$

where $\mathcal{H}$ is either 1 or

$$\frac{\overline{\theta}_v}{\theta_d(1 + r_v)}$$

Discuss the extent to which this represents a signficant difference in the downward acceleration produced by representative large values fof $\overline{r}_v = 20$ gm/kg and r_l of 5 gm/kg.

Due Thursday, February 16th.