Mean Fields Induced by Local Gravity-Wave Forcing in the Middle Atmosphere

XUN ZHU AND JAMES R. HOLTON

Department of Atmospheric Sciences, University of Washington, Seattle, WA 98195

(Manuscript received 29 May 1986, in final form 25 September 1986)

ABSTRACT

We examine the role of geostrophic adjustment in the middle atmosphere for given wave packet forcing by a three-dimensional hydrostatic model. The problem is solved directly so that solutions are expressed as convolution integrals of the Green’s function and the external forcing. It is shown that the induced fields consist of two different kinds of modes. One, produced by forcing vorticity only, is steady quasi-geostrophic flow; this is restricted to the forcing region. The other, produced by both forcing vorticity and forcing divergence, is oscillatory, and is in the form of gravity waves propagating out of the forcing region plus inertial oscillations in the forcing region. The scales and amplitudes of the induced gravity waves are determined by the forcing. For a typical example, a 200 km × 200 km gravity wave packet of momentum flux 0.5 N m⁻² absorbed in a layer of 5 km thickness centered near 18 km altitude, the gravity waves spread to a larger region ~ 100 km × 1000 km at a level 7 km above the forcing region. At that level the horizontal and vertical wavelengths of the induced waves are about 300 km and 7 km respectively, and the momentum flux is substantially reduced to 10⁻⁴ N m⁻². The wave parameters of such induced modes are consistent with the wave source parameters used in some general circulation models for the middle atmosphere. The results suggest that geostrophic adjustment processes may play an important role in specifying the gravity wave spectrum in the middle atmosphere.

1. Introduction

A number of recent studies have shown that gravity waves, through their ability to transfer momentum and energy to regions of the atmosphere remote from their sources, play an essential role in the general circulation of the middle atmosphere (see Fritts, 1984, for a review and extensive bibliography). As first noted by Hodges (1967), the exponential increase with height of gravity wave amplitude may lead to local convective overturning in the middle atmosphere, resulting in turbulent diffusion and momentum deposition. In the lower altitudes where the exponential amplification mechanisms are not predominant, the critical level mechanism can also cause some gravity waves to break (e.g., Geller et al., 1975; Yamanaka and Tanaka, 1984). Several workers (e.g., Walterscheid, 1984; Dunkerton and Fritts, 1984) have demonstrated the occurrence of such convective and/or dynamical wave breaking in simple gravity wave models.

Lindzen (1981) proposed a linear gravity wave saturation theory to quantitatively parameterize wave drag and diffusion induced by gravity wave breaking. His parameterization has been used in several studies of the circulation of the mesosphere (Holton, 1982; 1983; Schoeberl and Strobel, 1984; Holton and Zhu, 1984; Hunt, 1986). In all of these studies the characteristics of the gravity wave sources are specified arbitrarily, generally with the objective of modeling waves that achieve breaking amplitude in the mesosphere.

There is now renewed interest in the possible role of gravity wave breaking in the stratosphere. Although Lindzen (1984) showed that on a global basis the gravity waves produced by topography and shear collapse in the troposphere do not have sufficient amplitude to break until they reach the mesosphere, wave amplitudes may locally be much larger and the breaking level much lower. As a first approximation, assuming that the damping vanishes, the breaking height \( z_0 \) is given by (Holton and Zhu, 1984)

\[
z_0 = z_{00} + H \ln \left( \frac{(\bar{u} - c)^3 \rho_0 k}{2N M_0} \right),
\]

where \( H \) is the scale height and \( \rho_0 \) is the air density at the level \( z_{00} \),

\[
M_0 = \frac{\rho_0 k \Phi_0}{2N(\bar{u} - c)},
\]

is the momentum flux, and \( \Phi_0 \) is the perturbed geopotential at \( z_{00} \). Based on observational results for waves over the Rocky mountains (Lilly and Kennedy, 1973) we set \( |M_0| = 0.8 \) N m⁻², \( k = 2\pi/35 \) km, \( H = 7 \) km, and \( N = 0.02 \) s⁻¹. For convenience in our calculations we also set

\[
|\bar{u} - c| = 50 \exp \left( - \left( \frac{z - 10 000}{H} \right)^2 \right),
\]

\[
0 \leq z \leq 20 000 \text{ m, } [\text{m s}^{-1}].
\]

With these assumptions the breaking height is about 18.2 km, which is qualitatively consistent with the ob-
served strong turbulence layer near 16 km in their observations.

The above calculations show that wave breaking may occur in the stratosphere, and even near the tropopause. As noted by Lindzen (1985), the tendency for small phase-speed gravity waves to break in the stratosphere provides a constraint on the maximum wave amplitude in the mesosphere. Tanaka and Yamanaka (1985) suggested that the drag forces induced by breaking of mountain waves could have a crucial effect on the circulation of the stratosphere, while 0.1% of 0.5 N m⁻² is enough to simulate adequately the weak wind layer near the mesopause. Most of these works have postulated monochromatic waves. Observational evidence suggests, however, that there may be a saturated spectrum of gravity waves throughout the atmosphere (Smith et al., 1986) and that the spectrum broadens with height. A similar tendency for selective trapping of the smaller scales appears in the numerical model of mountain wave propagation, (Schoeberl, 1985).

In this paper we will show that the process of geostrophic adjustment to the local forcing could explain both transitions of wave amplitude and scale, i.e., how gravity waves of strong momentum flux in the troposphere reach the mesosphere with only a tiny momentum flux but with much larger horizontal wavelength and extent.

2. Model and solution

a. Local wave forcing induced mean flow equations

Since most mesoscale gravity waves in the troposphere produced by orography (lee waves) and wind shear have horizontal wavelengths less than 100 km, and confined to limited regions (Atkinson, 1981), the rectified effects of such waves may also be confined locally in space. Thus, it is useful to define a "mean flow" consisting of a local average over a wavelength of gravity wave. It is well known that the mean flow in the lower stratosphere can depart significantly from a symmetric polar vortex. This fact implies that the mean meridional velocity could have the same magnitude as the zonal mean velocity. We thus should have the following mean flow equations (Holton, 1975)

\[
\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = f \tilde{v} = -\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{v}^2}{\partial y} - \frac{1}{\rho_0 \rho_0} \frac{\partial}{\partial z} \rho \tilde{u} \tilde{w}, \tag{4a}
\]

\[
\frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} = f \tilde{u} = -\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{u}^2}{\partial x} - \frac{1}{\rho_0 \rho_0} \frac{\partial}{\partial z} \rho \tilde{v} \tilde{w}, \tag{4b}
\]
Furthermore, we assume that the induced flows are also small thus the nonlinear effect is neglected. From (4)-(10) we may get the following linear equations for induced flow as a first approximation

\[
\frac{\partial \bar{u}}{\partial t} + U \frac{\partial \bar{u}}{\partial x} + V \frac{\partial \bar{u}}{\partial y} - f_0 = -\frac{\partial \phi}{\partial x} - \frac{1}{\rho_0} \nabla_3 \cdot (kAe_\phi),
\]

\[
\frac{\partial \bar{v}}{\partial t} + U \frac{\partial \bar{v}}{\partial x} + V \frac{\partial \bar{v}}{\partial y} + f_0 = -\frac{\partial \phi}{\partial y} - \frac{1}{\rho_0} \nabla_3 \cdot (LAc_\phi),
\]

\[
\frac{\partial \phi_x}{\partial t} + U \frac{\partial \phi_x}{\partial x} + V \frac{\partial \phi_x}{\partial y} + N^2w = 0,
\]

\[
\frac{\partial \phi_z}{\partial t} + \frac{\partial \phi_z}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0w) = 0,
\]

in which we have replaced \((\bar{u}, \bar{v})\) by \((U, V)\) in the expressions of eddy flux terms.

For the convenience of calculation we introduce the Galilean transformation

\[
x^* = x - Ut, \quad y^* = y - Vt, \quad z^* = z, \quad t^* = t,
\]

which gives

\[
\frac{\partial}{\partial t^*} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y}.
\]

Now the equations (11a,b,c,d) become

\[
\frac{\partial \bar{u}}{\partial t} - f_0 = -\frac{\partial \phi_x}{\partial x} + F_x,
\]

\[
\frac{\partial \bar{v}}{\partial t} + f_0 = -\frac{\partial \phi_z}{\partial y} + F_z,
\]

\[
\frac{\partial \phi_x}{\partial x} + N^2w = 0,
\]

\[
\frac{\partial \phi_z}{\partial y} + \left(\frac{1}{\rho_0} \frac{\partial}{\partial z} - \frac{1}{H}\right)w = 0,
\]

where

\[
\mathbf{F} = (F_x, F_y) = \left(-\frac{1}{\rho_0} \nabla_3 \cdot kAe_\phi, -\frac{1}{\rho_0} \nabla_3 \cdot LAc_\phi\right).
\]

and the asterisks have been dropped in (14)-(15). Thus, without loss of generality we can assume that the basic flow is zero. This is similar to Miyahara’s (1985) mean flow model.

The model (14) describes the geostrophic adjustment process for a stratified fluid under external forcing \(\mathbf{F}\). Similar problems have been investigated very early in oceanic dynamics for the oceanic circulations driven by wind stress force (Veronis, 1956; Pollard, 1970). However, most studies in the atmosphere usually treat the adjustment problems for given initial conditions (Blumen, 1972). Such kinds of problems correspond mathematically to an initial impulsive forcing. A recent study by Miyahara (1985) was concerned with such a problem in the atmosphere. However, he filtered the inertio-gravity waves but kept the Rossby waves by arguing that the time scale (\(\sim 12\) h) is so short that geostrophic adjustment does not act. We think that his treatment is inappropriate for he missed the geostrophic adjustment process, which is important in some cases.

As the forcing \(\mathbf{F}\) vanishes the dispersion relations for the normal mode solutions are

\[
\omega^2 = 0, \quad (\omega^2, 3)^2 = f^2 + \frac{N^2(k^* + l^*)^2}{m^* + \frac{1}{4H^2}},
\]

where \(\omega^2\) and \(k^* = (k^*, l^*, m^*)\) are wave frequencies and wavenumber.

b. Solution by Green’s function

Since we are restricting the forcing mainly to a small region, for simplicity we consider the problem on an \(f\)-plane with constant Brunt–Väisälä frequency \((N)\). Under these conditions we can express the solution analytically in the form of a Green’s function.

From (14) we can form the following equations governing geopotential \(\phi\), vorticity \(\zeta\) and divergence \(\delta\):

\[
\mathbf{L}\phi = N^2F_s + fN^2 \int_0^t \mathbf{F}_d dr,
\]

\[
\mathbf{L}\zeta = L_x \left(\frac{\partial F_x}{\partial t} - fF_x\right) + N^2 \int_0^t \nabla^2 F_d dr,
\]

\[
\mathbf{L}\delta = L_x \left(\frac{\partial F_x}{\partial t} + fF_y\right),
\]

where

\[
\zeta = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \nabla^2 \psi, \quad \delta = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = -\nabla^2 \chi
\]

are vorticity and divergence respectively,

\[
\mathbf{L} = \left(\frac{\partial^2}{\partial t^2} + f^2\right)L_t + N^2\nabla^2, \quad L_z = \frac{\partial}{\partial z} \left(\frac{1}{\rho_0} \frac{\partial}{\partial z} - \frac{1}{H}\right)
\]

are linear differential operators, and

\[
F_t = \frac{\partial F_x}{\partial t} - \frac{\partial F_x}{\partial y}, \quad F_s = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}
\]

are external vorticity and divergence forcings respectively. The \(\psi\) and \(\chi\), introduced in (18a,b), are the non-divergent stream function and irrotational velocity potential, respectively. Induced horizontal velocity can be expressed by \(\psi\) and \(\chi\) as

\[
(u, v) = \left(-\frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y} \frac{\partial \chi}{\partial x}\right).
\]

It is known that the elementary solution of linear operator \(\mathbf{L}\) may be represented as a superposition of hydrostatic inertio–gravity wave modes and a steady mode of geostrophic flow (Dickinson, 1969; Blumen,
1972). Dispersion relations for these modes are given by (16). The solutions of (17) may also be considered as summed responses to the separated forcing terms given by the right-hand sides of these equations. The divergence (vorticity) forcing term may induce a vorticity (divergence) field through the second terms on the right-hand sides of (17b) or (17c) by the Coriolis effect. However, we will see that only gravity wave modes are included in the divergence field. On the other hand, it is the time integral of the vorticity forcing terms in (17a) and (17b) that induces the steady geostrophic mode.

It is well known that potential vorticity is a conservative quantity in geostrophic adjustment problems for given initial conditions (Gill, 1982). In the present problem we have the following generalized result

$$\frac{\partial \xi}{\partial t} = F \cdot \tau,$$

or

$$\xi = \int_0^t F \cdot \tau dt,$$

where

$$\xi = \xi + \frac{\int \tau}{N^2 L_x \phi}$$

is induced potential vorticity. Equation (23) shows that potential vorticity is a steady mode and only vorticity forcing makes a contribution to it. Solving

$$\left( \nabla^2 + \frac{\tau^2}{N^2 L_x \phi} \right) \Psi_g = \xi$$

for $\Psi_g$ gives the geostrophic streamfunction, which is both hydrostatic and geostrophic (Blumen, 1972).

Conservation of (initial) potential vorticity corresponds here to

$$F(x, t) = F(x, 0) \delta(t - t^0),$$

which gives

$$\xi(x, t) = F(x, 0) = \xi(x, 0),$$

in which the $\delta$ function represents an impulsive source of unit intensity (Dickinson, 1969).

Solutions of (17) can be expressed as convolution integrals of the right-hand forcing terms with a Green's function $G(x, t; x', t')$

$$\text{Solutions} = \int_0^t dt' \int dx' \text{Forcing}(x', t')G(x, t; x', t'),$$

where the Green's function is determined by

$$LG(x, t; x', t') = \delta(x - x')\delta(t - t').$$

For simplicity, we neglect the reflection of waves from the surface and consider the Green's function in free space. Following the method of Dickinson (1969) we obtain the Green's function

$$G(x, t; x', t') = -\frac{H(t - t' - \frac{r}{2HN} \exp[(z - z')/2H]}{4\pi Nr} \times J_0 \left[ \frac{r^2 + N^2(z - z')^2}{r^2} \right]^{1/2} \left[ (t - t')^2 - \frac{r^2}{4H^2N^2} \right]^{1/2},$$

where $r^2 = (x - x')^2 + (y - y')^2$.

3. Responses to point sources

In order to examine the responses to different forcing mechanisms we first consider the solution of (17a) for the following simple point sources.

Case A: A point impulsive forcing of unit divergence and vorticity at $x = 0$, $y = 0$, $z = z_0$ and $t = 0$:

$$F_\phi = F_\Omega = \delta(x) \delta(y) \delta(z - z_0) \delta(t).$$

Substituting (31) into (28) for (17a) we get the geopotential response

$$\phi_A(x, y, z, t) = N^2G(x, y, z, t; 0, 0, z_0, 0)$$

$$+ fN^2 \int_0^t G(x, y, z, t; 0, 0, z_0, t') dt'.$$

We see from (32) that the contribution from divergence forcing is a Green's function solution. On the other hand, the contribution from vorticity forcing is an integral of the Green's function. In order to see the differences between these two forcing mechanisms we specify $r = r_0 = 2HN/f$ and $z = z_0$, and examine the solution as a function of time. We write

$$\phi_A|_{r = 2HN/f} = \phi_A^1 + \phi_A^2,$$

where the superscripts 1 and 2 denote the responses to the divergence and vorticity forcings, respectively. Substituting (30) into (32) gives

$$\phi_A^1 = \frac{-f}{8\pi H} J_0 [(f)^2 - 1]^{1/2},$$

$$\phi_A^2 = \frac{-f}{8\pi H} \int_0^t J_0 [(s^2 - 1)^{1/2}] ds.$$
\[ \phi_{r}(x, y, z, t) = \delta N^{2} \int_{0}^{\infty} G(x, y, z, \tau; 0, 0, z, 0) d\tau \]
\[ + \left[ N^{2}G(x, y, z, t; 0, 0, z, 0) \right. \]
\[ \left. - \delta \int_{0}^{\infty} G(x, y, z, \tau; 0, 0, z, 0) d\tau \right], \quad (35) \]
i.e., the solution is the superposition of a steady mode and oscillating wave modes. The other curve shown in the Fig. 2 is the geopotential response at \( r = r_{0} \) and \( z = z_{0} - H \). We see that both the steady component and amplitude of oscillation are reduced mainly by the density effect. The period is reduced too. The vertical structure of point source responses may be seen by presenting phase contour lines of the Green's function in \( r-z \) and \( t-z \) sections (Fig. 3). The solid lines in Fig. 3 can be considered as wave fronts at a given time \((t = 5/\tau)\). The fronts propagate toward the level where the point source is located (shown by arrows). The dashed lines in the figure show that the periods decrease away from that level.

Case B: A point forcing of unit divergence and vorticity at \( x = 0, y = 0, \) and \( z = z_{0} \). The forcing lasts uniformly for the time interval \( T \). We thus have
\[ F_{s} = F_{t} = \begin{cases} \frac{1}{T} \delta(x) \delta(y) \delta(z - z_{0}), & 0 < t \leq T \\ 0, & t > T. \end{cases} \quad (36) \]

Note that case A is the limiting case of case B as \( T \) approaches zero. Calculating the solution under the same conditions as in A gives
\[ \phi_{\delta} = \begin{cases} \frac{-1}{8 \pi HT} \int_{1}^{\eta} J_{0}(\sqrt{s^{2} - 1}) ds, & 1 < \eta < fT + 1, \\
\frac{-1}{8 \pi HT} \int_{1}^{\eta} J_{0}(\sqrt{s^{2} - 1}) ds, & fT > 1, \end{cases} \quad (37a) \]
\[ \phi_{\delta} = \begin{cases} \frac{-1}{8 \pi HT} \int_{1}^{\eta} (fT - s) J_{0}(\sqrt{s^{2} - 1}) ds, & 1 < \eta \leq fT + 1, \\
\frac{-1}{8 \pi HT} \int_{1}^{\eta} (fT - s) J_{0}(\sqrt{s^{2} - 1}) ds, & fT > 1. \end{cases} \quad (37b) \]
4. Fields induced by local wave forcing

Local forcing by a wave packet might be a significant factor in the momentum budget of upper troposphere and low stratosphere (Lilly, 1972, Baldwin et al., 1985). The forcing of the gravity waves, which is parameterized by (5) as wave action flux convergence, could be mainly caused by total breaking of the wave into turbulence, and/or near critical level damping where the vertical group velocity is substantially reduced (Lindzen, 1985; Tanaka and Yamanaka, 1985). Thus, only the convergence of wave action flux along vertical group velocity makes a major contribution to the forcing. For convenience we specify the local forcing in the following form

\[
F_x = \begin{cases} 
F_0 \mu \left( \frac{x - c_x t - x_0}{W_x} \right) \mu \left( \frac{y - y_0}{W_y} \right) \mu \left( \frac{z - z_0}{W_z} \right), & 0 < t \leq T \\
0, & t > T
\end{cases}
\]

\[
F_y = 0,
\]

where \(c_x\) is the Doppler-shifted group velocity in the \(x\) direction, \(W_x, W_y\) and \(W_z\) are the forcing scales in \(x, y\), and \(z\) directions, respectively, and

\[
\mu(x) = \begin{cases} 
(256 x^4 (1 - x)^4, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases}
\]

Since the function given by (39) is continuous everywhere until the third derivative it is easy to calculate the solution of (17) numerically. The relation between momentum transferred to the mean flow \(M\) and \(F_x\) at the forcing center is

\[
M = \rho_m F_0 W_x \int_0^1 e^{-W_y H \left( \eta - 1/2 \right)} \mu(\eta) d\eta 
\]

\[
\approx \frac{128 e^{1/2}}{315} \left[ 1 - \frac{W_x}{2H} \right] + \frac{1}{2} \frac{(W_x)^2}{(2H)^2} \rho_m F_0 W_x, \quad (40)
\]

where \(\rho_m\) is the air density at the maximum forcing level \((z_m = z_0 + W_y/2)\). We thus estimate that if a wave packet of 0.5 N m \(^{-2}\) breaks at 18 km and its momentum flux is used entirely to accelerate a layer of \(\sim 5\) km it would produce a maximum local mean flow acceleration \(\sim 2.0 \times 10^{-3} \text{ m s}^{-2} \sim 170 \text{ m s}^{-1/\text{day}}\).

Substituting the forcing expression (38) and Green's function (30) into (28) gives solutions for the induced fields. From the vorticity and divergence fields we can obtain the horizontal velocity by solving Poisson's equations (18a,b) for stream function and velocity potential. We use the Gaussian quadrature formula to calculate the three dimensional space integral in (28) and use successive over-relaxation (SOR) method to solve the Poisson's equations. The time integral is calculated by the trapezoidal formula with time interval of 15 min.

The parameters used in the model are specified by setting \(H = 7\) km, \(f = 10^{-4} \text{ s}^{-1}\), \(N = 0.02 \text{ s}^{-1}\), and \(c_x = 10 \text{ m s}^{-1}\). Without losing generality we set \(x_0 = -W_y/2\), \(y_0 = -W_y/2\), \(z_0 = z_m - W_y/2\), which means that the maximum forcing is located at \(x = c_x t, y = 0\) and \(z = z_m\).

We first calculate the mean fields induced by forcing for the following parameters:

\[
F_0 = 0.002 \text{ m s}^{-2}, \quad W_x = W_y = 200 \text{ km,} \quad W_z = 5 \text{ km,} \quad T = 1 \text{ h,} \quad c_x = 10 \text{ m s}^{-1}.
\]

The forcing specified by (41) can be considered as due to a gravity wave packet of \(\sim 35\) km horizontal wavelength and 10 m s \(^{-1}\) Doppler-shifted group velocity which has been substantially absorbed within a cycle at \(\sim 18\) km. Figure 5 shows the calculated geopotential, stream function, and velocity-potential fields on the horizontal level \(z = z_m\) at the time \(t = 1\) h. We see that the induced fields are dipole patterns for this particular local forcing, which is supposed to represent forcing by a breaking gravity wave packet. Both the stream function and velocity-potential dipoles corresponding
to vorticity and divergence fields will produce an eastward flow in the forcing center. The dipole patterns are slightly rotated clockwise with respect to the forcing patterns. Comparing the geopotential field $\phi$ with stream function $\psi$ and velocity potential $\chi$ shows that the flow is strongly ageostrophic and essentially three-dimensional. The induced horizontal velocity is shown in Fig. 6. We see that significant flow is limited to the forcing region. The maximum eastward velocity reaches 2.6 m $s^{-1}$ at the forcing center.

From the equations (17a,b,c) we know that the steady part of the forcing effect is included in the geopotential and vorticity fields. In order to see how the induced flow changes with time we next calculate the vorticity, geopotential and divergence at $x = y = 50$ km, the forcing parameters are chosen as before. The
curve C in Fig. 7 shows the vorticity at $z = z_m$. Curves A and B are the vorticities induced by the continuous part of the forcing within $0 < t < 1$ h and the jump of the forcing at $t = 0$ and $t = 1$ h, respectively. We see that the vorticity increases uniformly with the time as $t < 1$ h. After the forcing is switched off it oscillates slightly about a steady value. Since the problem is linear we expect the induced vorticity to increase continuously while the forcing is acting; the induced zonal velocity would also increase steadily with time. Note that the forcing specified by the wave packet here is moving relative to the mean zonal flow. So, if we examine the responses locally, the maximum forcing time scale is the ratio of the horizontal scale of the wave packet to the Doppler-shifted group velocity. In the present case we have a maximum forcing time $\sim 200$ km/10 m s$^{-1} = 5.6$ h. Five hours of continuous forcing may induce a mean zonal flow of $\sim 10$ m s$^{-1}$. However, as the result of the change in the Doppler shifted frequency, forcing by the wave packet would also change, even if the wave sources were kept constant. Complete studies require coupled models including interactions between mean fields and forcing parameters.

The $t-z$ sections of geopotential and divergence fields are shown in Fig. 8. We see that most ageostrophic potential energy is radiated out of the forcing region just $1$ h after the forcing is switched off. The geopotential field then is in quasi-geostrophic balance ($\beta/\nu \ll 1$). Distinct propagation of phase toward the forcing region, which shows in the divergence field, confirms that gravity waves are induced in the forcing region. The density effect makes the perturbation asymmetric about the plane $z = z_m$. The wave propagates out of the forcing region with period $\sim 5$ h, whereas in the forcing region the oscillations tend to be inertial.

Figures 9 and 10 show the $x-z$ sections of geopotential at $y = 0$ with the same forcing parameters for $t = 4$ and $6$ h, respectively. The figures show clearly that the wave energy is spread to a much larger space than the forcing region as waves propagate out of the region. Comparing Figs. 9 and 10 verifies that there is downward phase propagation and upward energy propagation. Horizontal and vertical wavelengths at $z = z_m + H, x = \pm 300$ km are about $300$ km and $7$ km, respectively. Since the forcing was moving towards positive $x$, the wavenumber and the amplitude of the induced waves downstream (positive $x$) are slightly greater than those upstream. From dispersion relation (16) we can derive a period of $\sim 4$ h and horizontal and vertical group velocity of $\sim 25$ and 0.6 m s$^{-1}$, respectively. These are consistent with the phase relation between Figs. 9 and 10. If we assume that $z_m = 18$ km and estimate momentum flux ($\mathcal{M}$) at the level $z = z_m$.
The density effect of the atmosphere tends to make
gopotential perturbations increase exponentially as a
wave propagates upward. However, dispersion and/or
diffusion in n (n > 0) dimensional space may decrease
the perturbation as the wave propagates out of the
source region (Zeng, 1981). It can be proved that the
gopotential perturbations affected by these two factors
will decrease with height for \( z < z_m + nH \) and increase
rapidly above the level \( z = z_m + nH \). Observational
study of inertia–gravity waves in the region 30–60 km
by Hirota and Niki (1985) shows that wave amplitude
significantly decays with height compared with \( \exp(z/2H) \) factored profile. The decay being rather uniform
with height suggests that wave dispersion and diffusion
to a larger space is one possible decay mechanism.

Next we calculate the mean fields by setting the forci-
ging parameter \( T = 3 \) h in (41). Figures 11 and 12 are
gopotential, streamfunction, velocity-potential fields
and horizontal velocity field at the level \( z = z_m \) and
the time \( t = 3 \) h. Comparing with Figs. 5 and 6 shows
that the vorticity field is almost tripled and a distinct
induced mean flow exists over a much larger area. This
difference is because the steady mode is included
mainly in the vorticity field. Geopotential and diver-
genesis amplitudes increase only slightly for they are ra-
diated out of the forcing region continuously by gravity
wave modes. The maximum zonal velocity in the figure
is 5 m s\(^{-1}\). Figure 13 gives an \( x-z \) section of geopotential
at \( y = 0 \) for \( t = 6 \) h for the same forcing parameters.
The result shows that the wavelengths of the induced
waves are the same as before. The maximum momen-
tum flux at \( z = z_m + H \) now reaches \( \sim 10^{-4} \) N m\(^{-2}\).
Since gravity waves continuously radiate energy out of
the forcing region we will not expect significant increase
of momentum flux as forcing time (T) increases. We
thus can regard \( 10^{-4} \) N m\(^{-2}\) as a typical momentum
flux of the induced gravity waves at that level.

\[ + H \text{ by (2) we find that } M \sim 10^{-3} \text{ N m}^{-2}, \text{ which is}
\text{ substantially smaller than the momentum flux carried}
\text{ by the original gravity wave packet below } z = z_m. \text{ Fig-
}\text{ ures 9 and 10 show that as the height increases hori-
}\text{ zontal and vertical wavelengths slowly increase at the}
\text{ same rates. From (16) we know that wave frequency}
\text{ will not change significantly with the height.}

**FIG. 11.** As in Fig. 5 except the forcing lasts 3 h.

**FIG. 12.** As in Fig. 6 except the forcing lasts 3 h. The maximum
induced zonal velocity at the forcing center is 5 m s\(^{-1}\).
The spectrum of the induced gravity waves seems determined by the scale of the forcing region. Figure 14 shows the $x-z$ section of geopotential at $y = 0$ for $t = 6$ h by setting the forcing parameter $T = 3$ h and $W_x = W_z = 400$ km in (41). Now the horizontal wavelength is almost doubled while the vertical wavelength is slightly increased to $\sim 10$ km at $z = z_m + H$. We expect that the induced steady mean flow would exist in a larger area than shown in the Fig. 12.

5. Summary and remarks

We have proposed a three-dimensional model to study geostrophic adjustment processes under external forcing. Solving Eq. (14) directly by using a Green’s function is an especially powerful way to obtain the fields within a domain much greater than the forcing region. Since we considered the forcing by a gravity wave packet that was absorbed in the middle atmosphere, the only important forcing terms were momentum flux convergence terms. We derived the solution for steady flow, which was simply a generalization of the potential vorticity conservation law. For a more general case when there is a thermal forcing, term $(J)$ on the right-hand side of (14c), Eq. (23) will be replaced by

$$\tilde{\zeta}_S = \int_0^\infty \left[ F_t + \frac{f}{N^2} \left( \frac{\partial}{\partial z} \frac{1}{H} \right) J \right] dr.$$

(42)

Induced quasi-geostrophic steady flows are restricted to the forced region. However, there always exists a nonzero Doppler-shifted group velocity which allows the forcing to move along with the mean flow. Continuing forcing may induce mean fields in an area much larger than that of the wave packets. Those fields affected by the $\beta$-effect may correspond to some Rossby waves in the middle atmosphere.

Geostrophic adjustment could be important in redistributing the spectra of the gravity waves as they propagate from troposphere to mesosphere. Calculated results show that induced gravity waves have horizontal and vertical wavelengths comparable to the horizontal and vertical scales of the forcing wave packets, which are usually much larger than the wavelengths of the waves responsible for the forcing. The calculated vertical wavelengths ($\sim 10$ km) and wave periods ($\sim 4$ h) in the stratosphere are consistent with the observational results in the corresponding region (Hirota and Niki, 1985). The induced gravity waves are spread over a conical space as they propagate out of the forcing region. In the forcing region the induced oscillatory motions primarily resemble inertial oscillations in the lower stratosphere in agreement with observational evidence (e.g., Thompson, 1978; Sidi and Barat, 1986) and the theoretical study of Yamanaka (1985). The momentum flux by the induced gravity waves is substantially reduced. For typical wave sources given by (41) (without specifying $T$) the parameters of the induced waves are comparable with the wave sources specified at the lower boundary located at the tropopause. Further observational and theoretical studies are required in order to assess the importance of geostrophic process in re-arranging the wave spectrum in the middle atmosphere.

Acknowledgments. The authors thank Dr. Manabu D. Yamanaka for lots of valuable suggestions. One anonymous reviewer offered his/her many helpful comments. This research was supported by the National Aeronautics and Space Administrations through Grant NAGW-662.

REFERENCES


Yamanaka, M. D., 1985: Inertial oscillation and symmetric motion induced in an inertio-gravity wave critical layer. 63, 715–737.