Lognormal Conventions for Aerosol Models

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Lognormal Definitions

Basic definition is in terms of the number distribution:

$$\frac{dN}{d\ln(r)} = \frac{N}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{\left[\ln(r) - \ln(r_{g0})\right]^2}{2\sigma^2}\right\} \tag{1}$$

This is the most common mathematical convention. However, the total number, N, and the geometric number mean radius, r_{g0}, are very poor indicators of mass and light scattering properties of a given size distribution. At the same time, the total volume, V, and the geometric volume mean radius, r_{g3}, are very good indicators of mass and light scattering. Therefore, it is generally worthwhile to report not only N and r_{g0}, but also V and r_{g3}. These can be calculated analytically as follows:

$$r_{g3} = r_{g0} \exp(3\sigma^2) \tag{2}$$

$$V = N \frac{4\pi}{3} r_{g0}{}^3 \exp\left(\frac{9\sigma^2}{2}\right) \tag{3}$$

For reference, the cross-sectional area ($A=\pi r^2$), can be found analytically from:

$$A = N\pi r_{g0}{}^2 \exp\left(\frac{4\sigma^2}{2}\right) \tag{4}$$

To convert from volume to number, (2) and (3) can be rearranged,

$$r_{g0} = r_{g3} \exp(-3\sigma^2) \tag{5}$$

$$N = V \frac{3}{4\pi} r_{g0}{}^{-3} \exp\left(-\frac{9\sigma^2}{2}\right) \tag{6}$$

Integration of lognormal distributions:

The integration scheme begins by calculating the increment of particle number in the ith size bin,

$$dN_i = \frac{N}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{\left[\ln(r_i) - \ln(r_{g0})\right]^2}{2\sigma^2}\right\} d\ln(r)_i \tag{7}$$

where r_i means the ith radius. For equal increments of r, calculate $d\ln(r)_i$ (that is, the ith increment of the log of radius) from,

$$d\ln(r)_i = \ln\left[(r_i + dr/2) / (r_i - dr/2)\right] \tag{8}$$

To calculate the integral of any quantity, two additional parameters are needed: the lower and upper limits of integration, r_{lo} and r_{up}, respectively. If these are not 0 and ∞, we actually calculate integrals over a truncated size distribution. An overhat "~" will be used to indicate integrals or averages over truncated size distributions. The numerical integrals of cross-sectional area and volume are given by,

$$\widetilde{A} = \sum_{r_{lo}}^{r_{up}} \pi r_i{}^2 dN_i \tag{9}$$

$$\widetilde{V} = \sum_{r_{lo}}^{r_{up}} \frac{4\pi}{3} r_i^3 dN_i \qquad (10)$$

Tables 1 shows the physical properties of four, unimodal lognormal size distributions, using the above conventions. The specified parameters are shown with single digit precision and the calculated parameters are shown with 5-digit precision.

Table 1. Physical properties

Aerosol Type	V	r_{g3}	σ	r_{g0}	N	r_{lo}	r_{up}	$\widetilde{V}$
	($\mu m^3/cm^3$)	(μm)		(μm)	($\#/cm^3$)	(μm)	(μm)	($\mu m^3/cm^3$)
1. seasalt	1	1.0	ln(2)	0.23661	2.0743	0.01	5.0	0.9899
2. "dust"	1	1.5	ln(2)	0.35491	0.61461	0.01	5.0	0.9588
3. clean pollution	1	0.2	ln(2)	0.047321	259.29	0.01	5.0	1.0000
4. dirty pollution	1	0.2	ln(2)	0.047321	259.29	0.01	5.0	1.0000

Further reading:
The logic of representing measured aerosol size distributions as combinations of lognormally distributed "modes" (nuclei, accumulation, and coarse) was laid out in a seminal paper by Whitby (1978). The calculation of optical properties for lognormal distributions was discussed by Willeke and Brockmann (1977), who show the logic of working with the *volume* (not the number) distribution and of reporting the volume normalized extinction. Both these papers (and many others written within the aerosol measurement community) express the width of the distribution with exp(σ), rather than σ, and denote this quantity, σ_g. If Eq. 1 were rewritten by substituting $\ln(\sigma_g)$ for σ, it would follow this alternate convention. Obviously, this is the convention I am used to, since I specified σ in Table 1 as ln(2), where 2 would be σ_g according to the alternate convention.

[Note: The σ_g convention provides better intuitive insight. Take the example herein, where σ=0.69 and σ_g=2.0. If you integrate the number distribution from $r_{g0}/2$ to $r_{g0}*2$, you will include 67% of the total number. Similarly, if you integrate the volume distribution from $r_{g3}/2$ to $r_{g3}*2$, you will include 67% of the volume. Thus, the number 2.0 is a very useful guide to the properties of the size distribution. The same cannot be said of the number 0.69.]

The properties of lognormal size distributions (using the convention in Eq. 1) are discussed in Remer et al. (1998) and Boucher, Schwartz, et al. (1998). The former does a good job of presenting both number and volume properties (as recommended herein); the latter provides an good prototype for the rigor and care required when intercomparing Mie calculations and radiative transfer calculations based on aerosol size distributions.

Cited references:

Boucher, O., Schwartz, S. E., et al., 1998: Intercomparison of models representing shortwave radiative forcing by sulfate aerosols, *J. Geophys. Res.*, **103**, 16979-16998.

Remer, L. A. and Kaufman, Y. J., 1998: Dynamic aerosol model: Urban/industrial aerosol, *J. Geophys. Res.*, **103**, 13859-13871.

Whitby, K. T., 1978: The physical characteristics of sulfur aerosols, *Atmos. Envir.*, **12**, 135-159.

Willeke, K. and Brockmann, J. E., 1977: Extinction coefficients for multimodal atmospheric particle size distributions, *Atmos. Envir.*, **11**, 995-999.

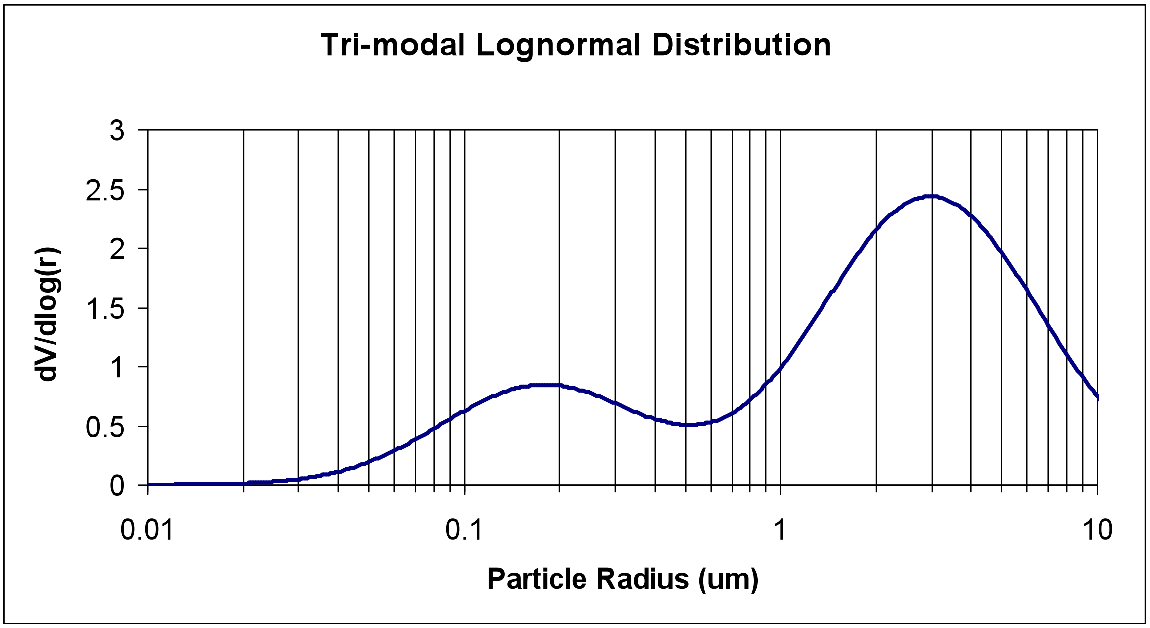
Problem:
Whitby, 1978, proposes the following parameters for a tri-modal, lognormal size distribution for an average clean continental background aerosol:

MODE	r_{g0} (μm)	σ_g	N ($\#/cm^3$)
Nuclei	0.008	ln(1.6)	1000
Accumulation	0.034	ln(2.1)	800
Coarse	0.46	ln(2.2)	0.72

i. Calculate r_{g3} for each mode.
ii. Calculate the volume ($\mu m^3/cm^3$) for each mode and for the total aerosol population.
iii. Plot the volume distribution. What radius is associated with the minimum between the two modes?

Answers:

MODE	r_{g3} (μm)	V ($\mu m^3/cm^3$)
Nuclei	0.016	0.006
Accumulation	0.18	1.6
Coarse	3.0	4.8
TOTAL		6.4



minimum is at about 0.5 microns particle radius