Introduction

Mathematical models used in many applications make use of the *smallness* of one thing compared to another. For instance, in elementary physics, we often neglect friction. In continuum mechanics, we rely on the smallness of an individual molecule compared to the bulk materials being modeled. The linear pendulum equation assumes the swing amplitude is infinitesimally small.

Smallness is a *nondimensional* concept; scale analysis and/or nondimensionalization of a physical equation is required to identify small parameters or terms. We will generally use ε to denote a small nondimensional parameter, so $\varepsilon \ll 1$. There is no universal rule about when a parameter ε or variable x is sufficiently 'small'; our methods often work well for values of ε or x of O(0.1) or less and gradually degrade above that, but this can vary greatly from problem to problem, as we'll see. A 'large' parameter or variable can be regarded as the reciprocal of a small parameter or variable.

Frequently, mathematical models explicitly involve small or large variables or parameters; systematically exploiting this is a powerful strategy which we will explore in this class.
Amath 568 is devoted to perturbative and approximate approaches to solving equations, with particular application to ordinary differential equations (ODEs):

- with small or large parameters
- for large or small x, or near a singularity in x.

The solutions will usually be in the form of *asymptotic series* which may or may not converge, but whose leading terms become increasingly good approximations to the exact solution as the parameter becomes more extreme or the reference point is approached. Such methods complement numerical solution methods, which often break down in such situations, and are representative of approximation strategies used across Applied Math. After taking this class, you will have a new awareness of small parameters and a toolbox of strategies for exploiting them.

Examples of ODE approximate solution

(1) Find the nonlinear correction to the period of a pendulum governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \varepsilon \ll 1, \frac{d\theta}{dt}(0) = 0.$$

(2) Explain 'parametric resonance', i.e. pumping a swing to make yourself go higher.

(3) Solve the steady-state diffusion-advection-reaction equation

$$\varepsilon y'' + 2y' + y^2 = 0, \quad y(0) = 0, y(1) = 1, 0 < \varepsilon \ll 1.$$

(4) How do the solution to Bessel's equation $y'' + x^{-1}y + k^2 y = 0$ behave as $x \to 0$? as $x \to \infty$? For finite x but large k?

(5) Find approximate eigenvalues $\lambda >> 1$ and corresponding eigenfunctions for Schrödinger's equation for the harmonic oscillator:

$$-y'' + x^2 y = \lambda y, \quad y(x) \to 0 \text{ as } x \to \pm\infty.$$

Perturbation Theory [Ref: B&O Ch. 7]

Perturbation theory is a collection of methods for solving problems involving a small parameter ε based on:

(1) Finding the solution $y_0(x)$ when $\varepsilon = 0$.

(2) Iteratively correcting it to deduce an approximate solution $y(x, \varepsilon)$ in the form of a series of terms that become successively smaller terms when $\varepsilon << 1$.

If each solution $y(x, \varepsilon)$ of the perturbed problem smoothly approaches a solution $y_0(x)$ of the unperturbed ($\varepsilon = 0$) problem as $\varepsilon \to 0$, this is called a *regular perturbation* of the $\varepsilon = 0$ problem. Otherwise it is called a *singular perturbation*.

Examples of regular perturbations:

Find the roots r of the quadratic polynomial

$$p(r;\varepsilon) = r^2 + \varepsilon r - 1 = 0, \quad \varepsilon \ll 1 \tag{R1}$$

Find the solution to the BVP:

$$y'' + \varepsilon y' = 1, \quad y(0) = 1, y(1) = 0, \varepsilon \ll 1. \tag{R2}$$

Find the solution to the BVP:

$$(1+\varepsilon)y'' + y' = 1, \quad y(0) = 1, y(1) = 0, \varepsilon \ll 1. \tag{R3}$$

In these examples, the perturbation does not change the degree of the polynomial or the order of the ODE. Thus, the number of solutions of the polynomial or ODE is not affected by the perturbation.

Examples of singular perturbations:

Find the roots r of the cubic polynomial

$$p(r;\varepsilon) = \varepsilon r^3 + r^2 - 1 = 0, \quad \varepsilon \ll 1 \tag{S1}$$

Find the solution to the BVP:

$$\varepsilon y'' + y' = 1, \quad y(0) = 1, y(1) = 0, \varepsilon \ll 1. \tag{S2}$$

In these examples, the perturbation adds a higher degree/order term to the equation. Thus, the number of solutions of the polynomial or ODE *is* affected by the perturbation. In this case, there is a part of the solution to the perturbed problem that has no counterpart in the unperturbed problem.