

WKB Theory: Waves in Slowly-Varying Media

Another important class of singular perturbation problems has the form

$$y'' - \varepsilon^{-2}Q(x)y = 0, \quad 0 < \varepsilon \ll 1. \quad (10.1)$$

where $Q(x)$ is some $O(1)$ smoothly varying function of a length coordinate x that has been nondimensionalized such that $O(1)$ variations of $Q(x)$ occur over $O(1)$ distances in x . The asymptotic approximation for its solution is called *WKB (Wentzel-Kramers-Brillouin) theory*. This class of equation commonly occurs in the study of linear waves propagating through spatially inhomogeneous media; we will study one such example in the next lecture. In that case, the small parameter ε will measure the ratio of the wavelength to the typical inhomogeneity lengthscale.

While (10.1) is certainly a singular perturbation problem (because the second order ODE reduces to the zeroth order ODE $y = 0$ if $\varepsilon = 0$) it is not immediately obvious how to start an asymptotic approximation to the solution in the limit of small ε . There are only two terms in (10.1) so it cannot be further simplified using a dominant balance between these terms. Instead, we recognize that we can asymptotically solve (10.1) *locally*, within a distance $O(\varepsilon)$ of some arbitrary reference point x_0 . In this range, $Q(x) = Q(x_0) + O(\varepsilon)$, and the approximate asymptotic general solution of the ODE has the form $y(x) \sim c_1 \exp\{Q^{1/2}(x_0)x / \varepsilon\} + c_2 \exp\{-Q^{1/2}(x_0)x / \varepsilon\}$. Where $Q(x_0) > 0$, this gives rapid exponential growth/decay; $Q(x_0) < 0$ gives rapid sinusoidal oscillations. The trick is how to generate a solution valid for all x which locally has these characteristics.

To do this, we focus on the function inside the exponent, which we saw above locally varies linearly with a slope of $O(\varepsilon^{-1})$, by setting

$$y(x) = \exp(S(x)) \quad (10.3)$$

Substituting into (10.1) yields

$$y'' = (S'e^S)' = (S'' + S'^2)e^S \quad \stackrel{r=S'}{\Rightarrow} \quad \underbrace{r'}_A + \underbrace{r^2}_B = \underbrace{Q(x)/\varepsilon^2}_C \quad (10.4)$$

We've replaced a 2nd-order linear ODE with a 1st-order nonlinear ODE (not necessarily a good trade!), but now we can do dominant balance on r . Our local argument implied that for x close to any reference point x_0 , $r(x) \approx \pm Q(x_0)^{1/2} / \varepsilon$. Thus we try the dominant balance $B \sim C \gg A$, which gives the *Eikonal equation*

$$r^2 \sim Q(x)/\varepsilon^2 \quad \Rightarrow \quad S' = r \sim r_0(x)/\varepsilon, \quad r_0 = \pm Q^{1/2}(x) \quad (10.5)$$

This dominant balance is consistent as long as $r'_0 \ll r_0^2 \Rightarrow \varepsilon |Q' / Q^{3/2}| \ll 1$. As long as $Q(x)$ is a smooth function of x , this is true except at *turning points* x_t where $Q(x_t) = 0$. The asymptotic behavior around a turning point is determined by approximating $Q(x) \approx (x - x_t)Q'(x_t)$, for which (10.1) turns into $y'' - \varepsilon^{-2}(x - x_t)Q'(x_t)y = 0$, a scaled version of *Airy's equation* which can be solved in terms of special functions called *Airy functions* that are related to Bessel functions. Using the asymptotic behavior of the Airy function for large arguments, we will later derive useful *connection formulas* connecting asymptotic solutions to (10.1) for $x > x_t$ with solutions valid for $x < x_t$.

We can develop an asymptotic series $r(x) \sim \varepsilon^{-1}(r_0 + \varepsilon^q r_1 + \varepsilon^{2q} r_2 \dots)$ and substitute into (10.4):

$$\varepsilon^{-1}(r'_0 + \varepsilon^q r'_1 + \varepsilon^{2q} r'_2 \dots) + \varepsilon^{-2}(r_0 + \varepsilon^q r_1 + \varepsilon^{2q} r_2 \dots)^2 = \varepsilon^{-2}Q(x)$$

The $O(\varepsilon^{-2})$ terms give the leading order solution (10.5). To balance the first neglected term A , which is $O(\varepsilon^{-1})$, we have to choose $q = 1$. Then, the next order in ε gives

$$O(\varepsilon^{-1}): \quad r'_0 + 2r_0 r_1 = 0 \quad \Rightarrow \quad r_1 = -r'_0 / 2r_0 = -Q' / 4Q$$

Putting the series back together

$$\begin{aligned} S' &= \pm \varepsilon^{-1} Q^{1/2}(x) - Q' / 4Q + O(\varepsilon) \\ \Rightarrow S^\pm(x) &= \pm \varepsilon^{-1} \int_{x_0}^x Q^{1/2}(\zeta) d\zeta - \frac{1}{4} \log |Q(x)| + O(\varepsilon) \\ \Rightarrow y^\pm(x) &= |Q(x)|^{-1/4} \exp \left\{ \pm \varepsilon^{-1} \int_{x_0}^x Q^{1/2}(\zeta) d\zeta \right\} \{1 + O(\varepsilon)\} \end{aligned} \quad (10.6)$$

This WKB approximation to the two linearly independent solutions of (10.1) is asymptotically accurate in the limit $\varepsilon \rightarrow 0$. If $Q > 0$, the two solutions $y^\pm(x)$ are the exponentially growing and decaying analogues of the two local solutions at the top of the previous page, with local e-folding scale $Q^{1/2}/\varepsilon$. If $Q < 0$, the two solutions can be regarded as right and left-propagating waves with spatially-varying local wavenumber $|Q|^{1/2}/\varepsilon$ and local amplitude $|Q|^{-1/4}$.

Application of WKB to dimensional problems with no explicit small parameter ε

Commonly, the WKB approximation is applied to a **dimensional** ODE

$$y'' - q(x)y = 0 \quad (10.1a)$$

of the form (10.1) with **no explicit small parameter**. Formally, the dominant balance approach starting with $y(x) = \exp\{S(x)\}$ will give the same solutions (but now with ϵ replaced by 1):

$$y^\pm(x) = |q(x)|^{-1/4} \exp\left\{\pm \int_{x_0}^x q^{1/2}(\zeta) d\zeta\right\} \left\{1 + O(|q'/q^{3/2}|)\right\} \quad \text{if } |q'/q^{3/2}| \ll 1 \quad (10.6a)$$

Here, the relative error estimate is based on the ratio between successive terms in the asymptotic series for S' . A common alternative form of (10.1a), especially in wave propagation problems, is to set $q(x) = -k^2(x)$.

$$y'' + k^2(x)y = 0 \quad (10.1b)$$

We interpret $k(x)$ as a local wavenumber for oscillations. In this notation, the two WKB solutions (10.6a) are:

$$y^\pm(x) = |k(x)|^{-1/2} \exp\left\{\pm i \int_{x_0}^x k(\zeta) d\zeta\right\} \left\{1 + O(|k'/k^2|)\right\} \quad \text{if } |k'/k^2| \ll 1 \quad (10.6b)$$

These solutions are still formally valid if $k^2(x) < 0$, in which case k is imaginary and we get decaying and growing exponentials. We can visualize the condition for validity of WKB as a ‘slowly varying medium approximation’ that the relative change of k in one wavelength $\lambda = 2\pi/k$ is small, i. e. that

$$\frac{\Delta k}{k} = \frac{\lambda k'}{k} \propto \frac{k'}{k^2} \ll 1 \quad (10.7)$$

(we have absorbed the order-1 factor 2π into the definition of ‘ \ll ’).

Sketch of WKB solutions

Sketches of the solutions are shown in Fig. 11.1. If $Q > 0$, the two solutions $y^\pm(x)$ are the exponentially growing and decaying analogues of the two local solutions at the top of page 10.2, with local e-folding scale $\epsilon/Q^{1/2}$. If $Q < 0$, the two solutions are right and left-propagating waves with spatially-varying local wavenumber $k = |Q|^{1/2}/\epsilon$ and local amplitude $|Q|^{-1/4} \propto |k|^{-1/2}$

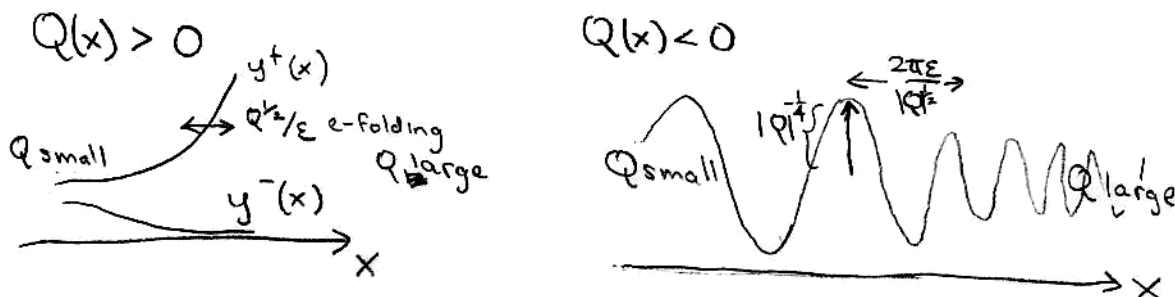


Fig. 11.1: Sketches of WKB solutions