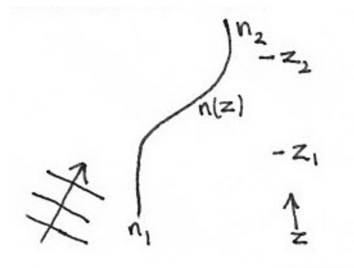
Example: Refraction of an oblique light wave by a change in refractive index $N(z)$



Consider a medium unbounded in x and z whose refractive index is constant for $z < z_1$, then smoothly increases from N_1 to N_2 in $z_1 < z < z_2 = z_1 + H$, and is constant for $z > z_2$. In this example, we use WKB (dimensional form) to calculate what happens to a short wavelength wave impinging on this refractive index gradient, as in the sketch to the right.

The electrical potential $u(x, z, t)$ obeys the 2D wave equation

$$u_{xx} + u_{zz} - \frac{N^2(z)}{c_0^2} u_{tt} = 0 \tag{11.1}$$

We consider a planar monochromatic light wave $u(x, z, t) = A \exp(ikx + im_1 x - i\omega t)$ in the region $z < z_1$ propagating obliquely up and right toward the refractive index gradient in the direction $\mathbf{k} = k\mathbf{i} + m_1\mathbf{j}$ of its vector wavenumber, which has an *incidence angle* $\theta_1 = \cot^{-1}(m_1/k)$ away from the vertical. To satisfy (11.1) for $z < z_1$, the wave frequency ω must obey the *dispersion relation*

$$k^2 + m_1^2 = N_1^2 \omega^2 / c^2 \tag{11.2}$$

What happens to the wave when it hits the gradient region? A solution to (11.1) of this structure can be sought in the form $u(x,z,t) = e^{i(kx-\omega t)} y(z)$:

$$0 = \frac{d^2 y}{dz^2} + \underbrace{\left(\frac{N^2(z)\omega^2}{c_0^2} - k^2 \right)}_{m^2(z)} y, \quad \text{where } y(z) \sim A \exp(im_1 z) \text{ as } z \to -\infty \tag{11.3}$$

Using the dispersion relation (11.2), we can write

$$m^2(z) = \frac{N^2(z)\omega^2}{c_0^2} - k^2 = \frac{N^2(z)}{N_1^2}\left(k^2 + m_1^2\right) - k^2 = k^2 \left(\frac{N^2(z)}{N_1^2 \sin^2 \theta_1} - 1 \right). \qquad (11.4).$$

When $kH << 1$, the wavelength will be short compared to the length scale of the refractive index change, so we can apply the formula (10.6b) (except with k, x replaced by m, z to obtain WKB approximations to the two linearly independent solutions:

$$y^{\pm}(z) = |m(z)|^{-1/2} \exp\left\{\pm i \int_{z_0}^{z} m(\zeta) d\zeta\right\}\left\{1 + O(m'/m^2)\right\}$$

Note that since $N^2(z)$ is O(1) and varies over a distance H, (11.4) implies that $m = O(k)$ and $m' = O(k/H)$, so $m'/m^2 = O(1/kH)$.

To match the form of the incident wave as $z \to -\infty$, we take A times the positive solution:

$$y(z) = A|m(z)|^{-1/2} \exp i\left\{\int_{z_0}^{z} m(\zeta) d\zeta\right\}\left\{1 + O(1/kH)\right\}, \quad kH >> 1 \tag{11.5}$$

which can be redimensionalized to the form

$$u(x,z,t) = A|m(z)|^{-1/2} \exp i\underbrace{\left\{kx + \int_{z_0}^{z} m(\zeta) d\zeta - \omega t\right\}}_{\text{wave phase } \psi(x,z,t)}\left\{1 + O(\frac{1}{kH})\right\} \tag{11.6}$$

Thus, the WKB asymptotic solution is a sinusoidal wave whose vertical wavenumber $m(z)$ changes as it moves across the refractive index gradient. This bends the direction of wave propagation (wave refraction). According to (11.4), m increases as N increases; in fact from (11.4) it is easy to show Snell's law that the angle of incidence of the wave obeys $N(z) \sin\theta = N_1 \sin\theta_1$.

Fig. 11.3 shows Re(u) for a specific example

$$N(z) = N_1 + (N_2 - N_1)\left\{\text{erf}(z) + 1\right\}/2,\ N_1 = 1,\ N_2 = 2 \tag{11.7}$$

in which N changes over roughly over a distance $H = 2$ between $z_1 = -1$ and $z_2 = 1$ with an incident wave with $k = m_1 = 2\pi$ so $kH = 4\pi$ '>>' 1, so WKB is quite accurate. Note the refraction of the wave toward the vertical as well as the amplitude reduction where N is larger.

If N increases with z as in this example, the WKB approximate solution consists purely of an upward-propagating wave. That is, there is no wave reflection in the WKB asymptotic limit that the wavelength is much shorter than the length scale H of the refractive index change. This can be contrasted to a step increase in refractive index (i.e. over a distance much *shorter* than the wavelength), which can be shown by direct solution of (11.3) to create a reflected wave of amplitude $(N_2 - N_1)/(N_2 + N_1)$ = 1/3 in this case, as well as a transmitted refracted wave. In general, partial reflection not predicted by the WKB approximation will occur wherever the medium varies over length scales shorter than a wavelength of the wave.

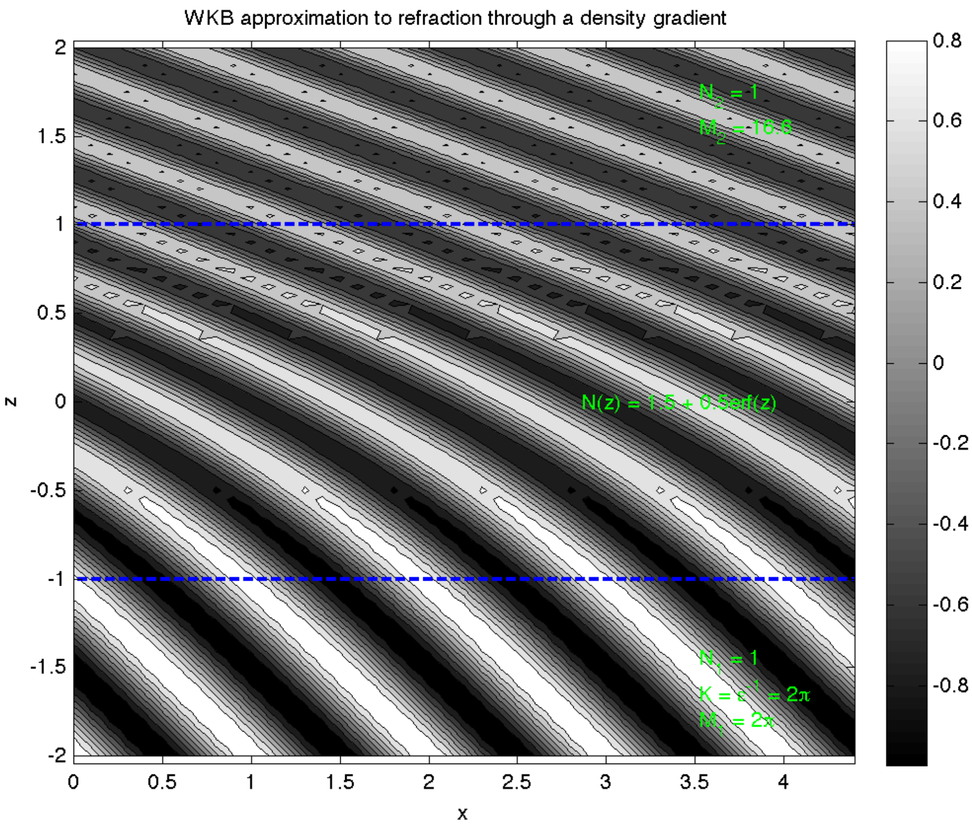


Fig. 11.3: WKB solution for wave refraction across a smooth increase in refractive index from 1 to 2 across the region between the blue dashed lines.