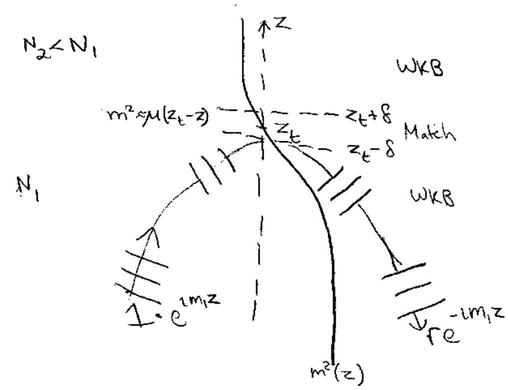
WKB and turning points

Consider the previous example of refraction of a light wave through a gradient in $N(z)$:

$$0 = y'' + m^2(z)y, \quad m^2(z) = k^2\left(\frac{N^2(z)}{N_1^2 \sin^2\theta_1} - 1\right), \quad \varepsilon = \frac{1}{kH} << 1, \quad y_{in}(z) \sim \exp(im_1 z) \text{ as } z \to -\infty, \tag{12.1}$$

but now with $N_2 < N_1$. If $N_2 > N_1 \sin\theta_1$, the wave will refract away from the vertical but the WKB solution is similar to the case $N_2 > N_1$. However, if $N_2 < N_1 \sin\theta_1$, there is a *turning point* z_t in the gradient region at which $m(z_t) = 0$. For $z > z_t$ $m^2(z) < 0$ (called an *evanescent* region) and the WKB solutions to (12.1) are growing and decaying exponentials. Hence the incident wave cannot propagate upward through the entire gradient region – can WKB asymptotics tell us what happens? Is the incident wave totally reflected? Partially reflected? Absorbed at the turning point?

Since WKB breaks down around the turning point, we introduce a thin *matching layer* $z_t + \delta > z > z_t - \delta$ with a half-width $\delta(\varepsilon)$ that goes to zero as $\varepsilon \to 0$. The idea is to find an approximate solution inside the matching layer and match its behavior to a WKB solution valid at the edges and outside the matching layer.



We'll choose the exact form of δ later, but we will assume $\delta << H$, so we can use the linear Taylor series approximation $m^2(z) \approx \mu(z_t - z)$ inside the matching layer. On the other hand, we'd like WKB asymptotics to be valid at the edges of the matching layer. The latter requires that where $|z - z_t| = \delta$,

$$1 \gg \left|\frac{m'}{m^2}\right| \propto \frac{\mu^{1/2} / |(z - z_t)|^{1/2}}{|\mu(z - z_t)|} = \frac{\mu^{1/2}}{|(z - z_t)|^{3/2}} \quad \Rightarrow \quad \delta \gg \mu^{-1/3}$$

Note that from (12.1), μ increases in proportion to k^2, so μ is larger for the WKB limit of short-wavelength waves. As the matching layer edge, we can pick any δ such that $\mu^{-1/3} << \delta << H$. We construct separate WKB solutions for the regions $z > z_t + \delta$ and $z < z_t - \delta$, then we match them using an inner solution valid across the matching layer.

Since the solution is excited by the specified incident wave, it must be bounded and any reflected or transmitted waves must be propagating outward from the gradient region. Hence we must choose WKB solutions to (12.1) such that

$$y(z)=\begin{cases} a|m(z)|^{-1/2}\exp\left(-\int_{z_t}^{z}\left[-m^2(\zeta)\right]^{1/2}d\zeta\right), & z>z_t+\delta \\ \underbrace{\left(\frac{m_1}{m(z)}\right)^{1/2}\exp\left(i\int_{z_t}^{z}m(\zeta)d\zeta\right)}_{\text{incident wave } m_1^{1/2}y^+(z;z_t)}+\underbrace{r\left(\frac{m_1}{m(z)}\right)^{1/2}\exp\left(-i\int_{z_t}^{z}m(\zeta)d\zeta\right)}_{\text{Reflected downward wave } rm_1^{1/2}y^-(z;z_t)\text{ of unknown amplitude } r}, & z<z_t-\delta \end{cases} \quad (12.2)$$

To find out a and the reflection coefficient r, we must find an inner solution that connects these two WKB solutions across the matching region. There, the ODE (12.1) can be approximated

$$0\approx y''+\mu(z_t-z)y, \quad |z-z_t|<\delta \quad (12.3)$$

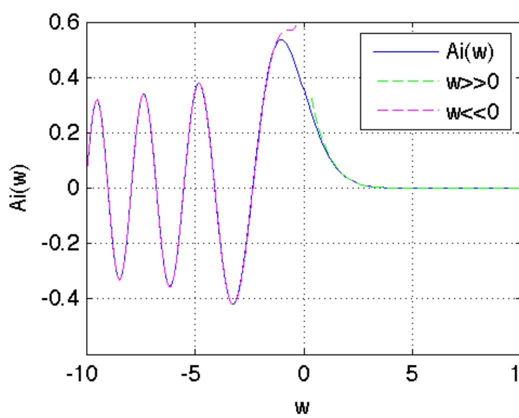
The constants can be removed by rescaling $w = \alpha(z - z_t)$. Sloppily regarding y now as a function of w rather than z to avoid introducing yet another variable, (12.3) becomes

$$0=\alpha^2\frac{d^2y}{dw^2}-\alpha^{-1}\mu wy, \quad |w|<\alpha\delta$$

Taking $\alpha=\mu^{1/3}$, this simplifies to *Airy's equation*

$$\frac{d^2y}{dw^2}=wy, \quad |w|<w_\delta=\mu^{1/3}\delta$$

The two linearly independent solutions are special functions called the *Airy functions* of the first and second kind, Ai(w) and Bi(w), which are related to Bessel functions. Ai(w) (see plot) is the solution that decays as $w \to +\infty$, the behavior we want.



Earlier, we argued that the matching region edge should satisfy $\delta \gg \mu^{-1/3}$. This is equivalent to taking the edge at $w_\delta \gg 1$. To match onto the WKB solution at this edge, we need to know how Ai(w) behaves for large w. One can show that the Fourier transform of the Airy function is $\exp(ik^3/3)$ and use 'stationary phase' asymptotics on the inverse Fourier transform integral (or Abramowitz and Stegun) to deduce the asymptotic behavior of Ai(w) for large w – this is a great exercise for a more advanced class:

$$Ai(w)=\begin{cases}\frac{1}{2\pi^{1/2}w^{1/4}}\exp\left(-\frac{2}{3}w^{3/2}\right), & w\gg 0\\ \frac{1}{\pi^{1/2}(-w)^{1/4}}\sin\left(\frac{2}{3}(-w)^{3/2}+\frac{\pi}{4}\right), & w\ll 0\end{cases} \quad (12.4)$$

The green and magenta curves in the above plot show these are great approximations to Ai(w) for w >>0.5 and << 0.5, respectively. In particular, they can be applied at the matching points $w = \pm w_\delta$.

At the edges of the matching region, the WKB solutions (12.2) can also be evaluated, approximating $m^2(z) \approx \mu(z_t - z)$ in the integrals (OK, since the integrals are from z_t to $z_t \pm\delta$). Starting with z_t -δ (or w = -w_δ):

$$\begin{aligned} y_{WKB}(z_t-\delta) &= \frac{m_1^{1/2}}{[\mu\delta]^{1/4}}\left\{\exp\left(i\int_{z_t}^{z_t-\delta}[\mu(z_t-\zeta)]^{1/2}d\zeta\right)+r\exp\left(-i\int_{z_t}^{z_t-\delta}[\mu(z_t-\zeta)]^{1/2}d\zeta\right)\right\} \\ &= \frac{m_1^{1/2}}{[\mu\alpha^{-1}w_\delta]^{1/4}}\left\{\exp\left(i\int_0^{-w_\delta}[-w]^{1/2}dw\right)+r\exp\left(-i\int_0^{-w_\delta}[-w]^{1/2}dw\right)\right\} \\ &= \frac{m_1^{1/2}}{\mu^{1/6}}w_\delta^{-1/4}\left\{\exp\left(-\frac{2}{3}iw_\delta^{3/2}\right)+r\exp\left(\frac{2}{3}iw_\delta^{3/2}\right)\right\} \end{aligned}$$

We match this to an 'inner' solution $y_{in}(z) = b$ Ai(w) of unknown amplitude b at w = -w_δ. From (12.4),

$$y_{in}(-w_\delta) = b\,\mathrm{Ai}(-w_\delta) \sim \frac{1}{2i\pi^{1/2}w_\delta^{1/4}}\left\{\exp\left(\frac{2}{3}iw_\delta^{3/2}+\frac{i\pi}{4}\right)-\exp\left(-\frac{2}{3}iw_\delta^{3/2}-\frac{i\pi}{4}\right)\right\}$$

$$\Rightarrow \quad b = \frac{2\pi^{1/2}m_1^{1/2}}{\mu^{1/6}}e^{-i\pi/4}, \quad r = e^{-i\pi/2}$$

The wave reflects perfectly with a phase shift of -π/2. A similar argument at z_t +δ (or $w = w_\delta$): (exercise for the reader) shows the coefficient of the evanescent WKB solution is $a = m_1^{1/2}e^{-i\pi/4}$.

This analysis can be encapsulated in the *turning point matching formula*

$$y \text{ decaying for } z > z_t \quad \Rightarrow \quad y_{WKB}(z) = c\left\{y^+(z;z_t) - iy^-(z;z_t)\right\} \text{ for } z < z_t \qquad (12.5)$$

A similar analysis for the case where $m^2(z)$<0 *below* the turning point implies:

$$y \text{ decaying for } z < z_t \quad \Rightarrow \quad y_{WKB}(z) = c\left\{y^+(z;z_t) + iy^-(z;z_t)\right\} \text{ for } z > z_t \qquad (12.6)$$

where $y^\pm(z;z_t) = |m(z)|^{-1/2}\exp\left\{\pm i\int_{z_t}^{z} m(\zeta)d\zeta\right\}$. These formulas are suitable boundary conditions at turning points for WKB solutions so that we don't have to redo the above analysis in other cases.

Fig. 12.3 shows the matched solution for the case

$$N(z) = N_1 + (N_2 - N_1)\left\{\mathrm{erf}(z)+1\right\}/2,\ N_1 = 2,\ N_2 = 1$$

with the same incident wave with $k = m_1 = 2\pi$ as in the previous lecture. Taking H = 2, this again corresponds to ε = $(kH)^{-1}$= $(4\pi)^{-1}$. There is a turning point at z_t = 0.15 with

$\mu = -d(m^2)/dz\big|_{z_t} = 0.77k^2$ so $w = 3.1k^2(z\text{-}z_t)$. We take as the edge of the matching region. The Matlab script `WKB_reflect.m` on the class web page plots the Airy solution for $w > -w_\delta$ ($z > -0.49$) and the WKB solution for $z < -0.49$, calculated by numerical quadrature of the phase integral. Note the smooth transition from exponential decay above to a standing wave made by interference of the incident and reflected wave below the gradient region, as well as the slightly increased amplitude near the turning point. The Airy and WKB solutions blend perfectly at the matching level $z = -0.49$, as the asymptotics imply. Technically, we should use the exponentially decaying WKB solution rather than the Airy function for $w > w_\delta = 2$ ($z > 0.81$), but by this point the Airy function has decayed so much that it is visually (even if not asymptotically) identical to the WKB solution.

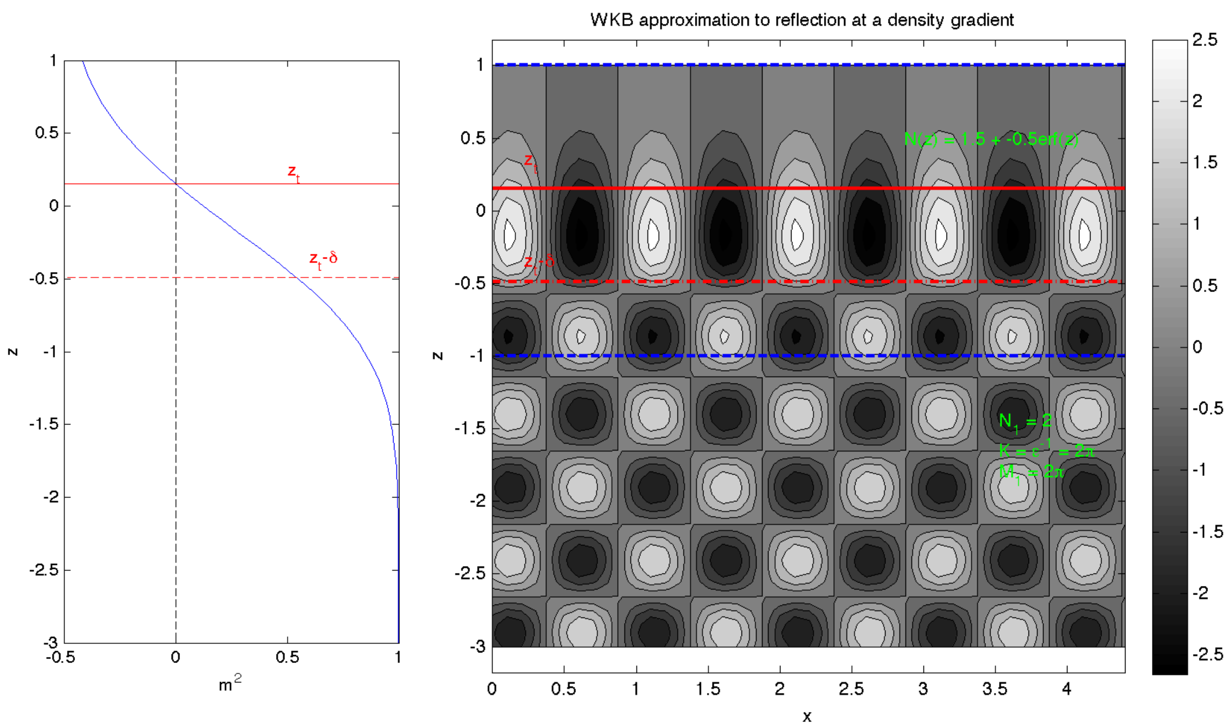


Fig. 12.3: WKB wave reflection example. A wave of horizontal wavenumber $\varepsilon^{-1} = 2\pi$ is incident at 45 degrees to a gradual decrease of refractive index from 2 to 1 above the blue dashed line. Left – Squared vertical wavenumber. Right – asymptotic solution using WKB below the dashed red line and the inner Airy solution above. Above the turning point z_t (solid red line) solution is evanescent.