Approximate solutions near singular points of ODEs

We turn to our second major topic of this class, which is to develop techniques for the approximate solution of ODEs near their *singular points*. This seemingly arcane topic gives us useful asymptotic approximations to solutions of many important ODEs, such as Bessel functions, Airy functions, error functions, etc. It is also closely related to our derivation of the WKB approximation.

Ordinary and singular points of ODEs

Consider the n'th order homogeneous linear ODE

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x)\ldots + p_1(x)y' + p_0(x)y = 0, \quad y^{(k)}(x) \equiv \frac{d^k y}{dx^k} \tag{14.1}$$

Assume that the $p_k(x)$ are defined for complex as well as real arguments x. The point $x_0 \neq \infty$ is an *ordinary point* of (14.1) if all $p_k(x)$, $k = 0, \ldots, n-1$ are analytic in a neighborhood of x_0 in the complex plane. Otherwise it is called a *singular point*.

Example: For Airy's equation $y'' - xy = 0$, $p_1(x) = 0$, $p_0(x) = -x$. Both are entire (analytic for all complex x) so all finite points x_0 are ordinary points of Airy's equation.

Taylor series for solutions around ordinary points of an ODE

Theorem (Fuchs 1866): Let R be the distance from x_0 to the nearest singularity of any $p_k(x)$ in the complex plane. Then all n linearly independent solutions of (8) are analytic and have convergent Taylor series (TS) expansions about x_0 in $|x - x_0| < R$.

Note that Taylor series are usually efficient approximants (i. e., a few leading terms give an accurate approximation to the solution) only near $x = x_0$, even if they converge for much larger $|x - x_0|$.

Example: Find TS solutions to Airy's equation around the ordinary point $x_0 = 0$. By the above theorem, these series will converge for all finite complex x. The form of the series is

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \underbrace{=}_{x_0=0} \sum_{m=0}^{\infty} a_m x^m$$

so
$$y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} \underbrace{=}_{\text{Set } n=m-2} \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

and
$$xy = \sum_{m=0}^{\infty} a_m x^{m+1} \underbrace{=}_{\text{Set } n=m+1} \sum_{n=1}^{\infty} a_{n-1}x^n$$

Thus
$$0 = y'' - xy = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n$$

Noting that unique Taylor series for 0 is $\sum_{n=0}^{\infty} 0x^n$ and matching powers of x:

x^0: $0 = 2\cdot 1\cdot a_2 \quad \Rightarrow a_2 = 0$

x^n, $n > 0$: $0 = (n+2)(n+1)a_{n+2} - a_{n-1} \quad \Rightarrow \quad a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$

The two coefficients a_0 and a_1 are arbitrary. These recursion formulas imply:

$$a_{3p} = \frac{a_{3p-3}}{3p(3p-1)} = \ldots = \underbrace{\frac{1}{3p(3p-1)(3p-3)(3p-4)\ldots 3\cdot 2}}_{b_p} a_0$$

$$a_{3p+1} = \frac{a_{3p-2}}{(3p+1)(3p)} = \ldots = \underbrace{\frac{1}{(3p+1)(3p)(3p-2)(3p-3)\ldots 4\cdot 3}}_{c_p} a_1$$

$$a_{3p+2} = \frac{a_{3p-1}}{(3p+2)(3p+1)} = \ldots = \frac{a_2}{(3p+2)(3p+1)(3p-1)(3p-2)\ldots 5\cdot 4} = 0$$

so
$$y(x) = a_0 \underbrace{\sum_{p=0}^{\infty} b_p x^{3p}}_{y_0(x)} + a_1 \underbrace{\sum_{p=0}^{\infty} c_p x^{3p+1}}_{y_1(x)}$$

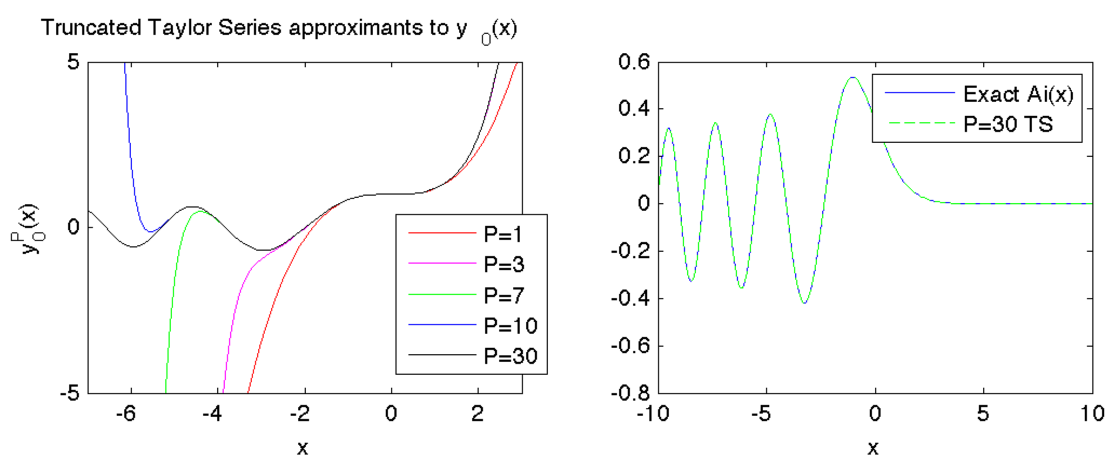
which is a linear combination of TS for two linearly independent solutions of Airy's equation, given by the series expansions $y_0(x)$ and $y_1(x)$. A ratio test (of terms $p+1$ to term p in either series) confirms that both series converge for all x, as our theorem asserted.

The Matlab script `airy_ts.m` (next page) calculates the truncated TS expansions

$$y_0^P(x) = \sum_{p=0}^{P} b_p x^{3p}, \quad y_1^P(x) = \sum_{p=0}^{P} c_p x^{3p+1}$$

for a user-chosen P and range $\{x\}$. `airy_plot.m` plots $y_0^P(x)$ over $-7 < x < 3$ for selected P's (figure on the bottom of next page). For $P = 10$, the TS is accurate for $|x| \le 5$ (i.e. adding further terms does not significantly change the sum); for $P = 30$ it is accurate for $|x| \le 10$. As expected, more terms are needed for accuracy at larger $|x|$. While 30 terms are impossible to visualize and would be a lot for using a hand calculator, even taking 100 terms of the TS is still quite computationally efficient for Matlab.

```
function [y0,y1] = airy_ts(x,P)

% Return the Taylor series about x0=0 for two independent solutions
% of Airy's equation y" - x*y = 0:
%
% y0(x) = sum_{p=0}^P b_p x^{3p}
% y1(x) = sum_{p=0}^P c_p x^{3p+1}
%
% x may be a vector of values.

  nx = length(x);

  % Start with term p=0
  bp = 1;
  cp = 1;
  x3p = ones(1,nx);
  x3p1= x;
  y0 = bp*x3p;
  y1 = cp*x3p1;

  % Add other terms using Airy TS recursion formulas

  x3 = x.^3;
  for p = 1:P
    bp = bp/(3*p * (3*p-1));
    cp = cp/((3*p+1) * 3*p);
    x3p = x3p.*x3;
    x3p1 = x3p1.*x3;
    y0 = y0 + bp*x3p;
    y1 = y1 + cp*x3p1;
  end
```



The Airy function we've been using for WKB turning point analysis must be a linear combination $\mathrm{Ai}(x) = a_0 y_0(x) + a_1 y_1(x)$. Noting Ai(0)=0.355 and Ai(10)=$1.1\times 10^{-10} \approx 0$, and using the $P = 30$ truncations, we can deduce that $a_0 = 0.355$ and $a_1 = -0.258$. The resulting $P = 30$ truncated TS for Ai(x) perfectly matches the Matlab-computed 'exact' value for $|x| \leq 10$, as seen in right plot above.