The asymptotic form of the two solutions as $x \to 0$ is $y_{1,2}(x) \sim x^{\nu,-\nu}$ unless ν = 0, when $y_1 \sim 1$ and $y_2 \sim \log(x)$. As is often true, the most relevant cases correspond to complications (ν = 0 ↔ double root; ν = *n* ↔ $\alpha_1 - \alpha_2 = 2n$ = integer). Regardless, one solution will always have the form $y_1(x) = y(x;\nu)$, where

$$y(x;\alpha) = x^\alpha \sum_{n=0}^{\infty} a_n(\alpha)x^n = \sum_{n=0}^{\infty} a_n(\alpha)x^{n+\alpha}, \tag{15.2}$$

We write *y*(*x*; α) as a function of α rather than immediately setting α = ν so that

(a) If $\alpha_1 - \alpha_2 = 2\nu$ is not an integer, we can derive $y_2(x)=y(x; -\nu)$.

(b) For the case of equal roots ν = 0, we can find $y_2(x) = \partial y(x;\alpha)/\partial\alpha|_{\alpha=\nu}$.

Now (15.2) implies

$$y'' = \sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2}$$

$$\frac{y'}{x} = \sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-2}$$

$$\left(1-\frac{\nu^2}{x^2}\right)y = \sum_{m=0}^{\infty} a_m x^{m+\alpha} - \nu^2\sum_{n=0}^{\infty} a_n x^{n+\alpha-2} \underset{m=n-2}{=} \sum_{n=2}^{\infty} a_{n-2}x^{n+\alpha-2} - \nu^2\sum_{n=0}^{\infty} a_n x^{n+\alpha-2}$$

Substituting into (BE) and collecting like powers of *x*,

$n = 0$ $(x^{\alpha-2})$: $\alpha(\alpha-1)+\alpha-\nu^2 = 0$ (the indicial eqn, automatically satisfied by α=ν.)

$n = 1$ $(x^{\alpha-1})$: $\left[(\alpha+1)\alpha+\alpha+1-\nu^2\right]a_1 = 0 \Rightarrow a_1 = 0$

$n \geq 2$ $(x^{\alpha+n-2})$: $[\underbrace{(\alpha+n)(\alpha+n-1)+\alpha+n}_{(\alpha+n)^2}-\nu^2]a_n = -a_{n-2} \Rightarrow a_n = -\dfrac{a_{n-2}}{(\alpha+n-\nu)(\alpha+n+\nu)}$

Note that an inconsistency is possible if $\alpha + n = \nu$, because the coefficient on the left hand side will be zero, and the right hand side has been determined in a previous iteration and may not be zero. This can't happen for the positive root $\alpha_1=\nu$, but for $\alpha_2=-\nu$, inconsistency can occur if $n = 2\nu = \alpha_1 - \alpha_2$. This is why we need to make an exception for the case that $\alpha_1 - \alpha_2$ is an integer.

Iterating, we obtain:

$$a_{2p}(\alpha) = \frac{(-1)^p a_0(\alpha)}{(\alpha+2p-\nu)(\alpha+2p-2-\nu)\ldots(\alpha+2-\nu)(\alpha+2p+\nu)(\alpha+2p-2+\nu)\ldots(\alpha+2+\nu)}$$

$$a_{2p+1}(\alpha) = 0$$

At this point, we take α = ν (the 'true' α), whence

$$a_{2p}(\nu) = \frac{(-1)^p a_0(\nu)}{(2p)(2p-2)...(2)(2p+2\nu)(2p-2+2\nu)...(2+2\nu)} = \frac{(-1)^p a_0(\nu)}{2^{2m} p!(p+\nu)...(1+\nu)}$$

Making use of the *gamma function* $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, for which $\Gamma(n+1) = n!$ and $\Gamma(z+1) = z\Gamma(z)$, we can write $(p+\nu)...(1+\nu) = \Gamma(\nu+p+1)/\Gamma(\nu+1)$. Hence

$$y_1(x) = a_0 \Gamma(\nu+1) 2^\nu \underbrace{\sum_{p=0}^{\infty} \frac{(-1)^p (x/2)^{2p+\nu}}{p!\Gamma(\nu+p+1)}}_{J_\nu(x)}$$

With appropriate normalization $a_0(\nu) = 2^{-\nu}/\Gamma(\nu+1)$, this is the Frobenius series for $J_\nu(x)$, the *Bessel function of order ν.*

The second ($\alpha_2 = -\nu$) solution

(1) If $\alpha_1 - \alpha_2 = 2\nu$ is noninteger, the second solution is $y_2(x) = y(x; -\nu) = J_{-\nu}(x)$.

(2) If $2\nu = 2P + 1$ is odd, this solution still works because the $n = 2P+1$ terms in the Frobenius series give $[(-\nu + 2P + 1)^2 - \nu^2] a_{2P+1} = -a_{2P-1} = 0$. Although the LHS is zero, which could have produced an inconsistency, the RHS is also zero since the odd coefficients are zero. Thus there is no inconsistency in taking $a_{2P+1}=0$.

(3) If $2\nu = 2P$ is even and positive, the second solution is too messy for in-class derivation.

(4) If $\nu = 0$, the second solution is

$$y_2(x) = \frac{\partial}{\partial \alpha} y(x;\alpha)\bigg|_{\alpha=0} = \sum_{p=0}^{\infty} \frac{\partial}{\partial \alpha}\left[a_{2p}(\alpha) x^{2p+\alpha}\right]_{\alpha=0}$$

$$= \sum_{p=0}^{\infty} \left[a_{2p}(\alpha) x^{2p+\alpha} \log x + \frac{\partial a_{2p}}{\partial \alpha} x^{2p+\alpha} \right]_{\alpha=0} = J_0(x) \log x + \sum_{p=0}^{\infty} b_p x^{2p}$$

where

$$b_{2p} = \frac{\partial a_{2p}}{\partial \alpha}\bigg|_{\alpha=0} = a_{2p} \frac{\partial}{\partial \alpha} \log a_{2p}(\alpha)\bigg|_{\alpha=0}$$

$$= a_{2p}(0) \frac{\partial}{\partial \alpha}\left[-2\sum_{k=0}^{p} \log(\alpha+2k)\right]_{\alpha=0} = a_{2p}(0)\left[-2\sum_{k=1}^{p} \frac{1}{\alpha+2k}\right]_{\alpha=0} = \frac{(-1)^{p+1}}{2^{2p}(p!)^2} H_p$$

and $H_p = \sum_{k=1}^{p} \frac{1}{k}$ is the p'th harmonic number. *The Bessel function of the second kind of order zero* is $Y_0(x)=\frac{2}{\pi}\left\{y_2(x)+(\gamma-\log 2)J_0(x)\right\}$, a linear combination of the two Frobenius solutions. Here $\gamma = \lim_{n\to\infty}\left\{H_n - \log n\right\} = 0.577\ldots$ is Euler's constant.

The functions $J_0(x)$ and $Y_0(x)$ are the functional analogues to cos x and sin x in cylindrical coordinates. They can be combined into the Hankel functions of order zero, $H_0^{(1)}(x) = J_0(x) + iY_0(x)$ and $H_0^{(2)}(x) = J_0(x) - iY_0(x)$, which function like exp($\pm ix$) and are useful for solving such problems as finding the structure of waves radiated from a long linear antenna.

The plot below, generated by Matlab script `bessel_plot.m` (class web page) shows the convergence of the Frobenius series for $J_0(x)$ and $Y_0(x)$ truncated after the x^{2P} term for various P. With P = 13, both series are quite accurate for $0 < x < 10$.