Approximate solution by dominant balance near an irregular singular point (B&O 3.4-3.5)

Near an irregular singular point x_0 of a linear homogeneous ODE, we can find an asymptotic series solution via the method of dominant balance. Most common ODEs have irregular singular points at $x_0 = \infty$, so this is a very powerful method for finding the asymptotic behavior of solutions of an ODE for large x. We will demonstrate the method on a 2nd order ODE

$$y'' + p_1(x)y' + p_0(x)y = 0\,, \tag{17.1}$$

but it can be applied to ODEs of other orders as well. It works as follows:

(1) As with WKB, set $y(x) = e^{S(x)}$, and derive a nonlinear ODE for S',

so (17.1) $\Rightarrow$ $$S'^2 + S'' + p_1(x)S' + p_0(x) = 0 \tag{17.2}$$

(2) We look for a consistent dominant balance on the LHS of (17.2) as $x \to x_0$, from which we deduce a leading-order asymptotic behavior $S_0(x)$ of the solutions Two key facts are

(a) Near an irregular singular point, we expect $y(x)$ to be more singular than at a regular singular point (since the singularity in the coefficients is stronger). Thus $y(x)$ should grow or decay faster than any power of $x - x_0$, so $S(x)$ should have a singularity stronger than O[log($x - x_0$)]. This is consistent with the behavior $S(x) = O[(x - x_0)^{-r}]$, $r > 0$, whence $S' = O[(x - x_0)^{-r-1}]$ and $S'' = O[(x - x_0)^{-r-2}]$. Thus $S''/S'^2 = O[(x - x_0)^{-r-2}/(x - x_0)^{-2r-2}] = O[(x - x_0)^{r}] \to 0$ as $x \to x_0$. Hence we can assume $S'' \ll S'^2$. This ensures that the leading order balance in (17.2) is a quadratic equation in S', which is sure to be solvable.

(b) Include only the leading asymptotic behavior of $p_1(x)$ and $p_0(x)$ to get the simplest balance.

(3) Now write $S' = S_0' + S_r'$, substitute into (17.2), and look for a new dominant balance for the residual S_r', whose asymptotic solution we call $S_1(x)$. In constructing the dominant balance note that $S_r' / S_0' \to 0$ as $x \to x_0$. Assuming that S_r' and S_0' go like powers of $x - x_0$, this also implies $S_r''/ S_0'' \to 0$.

(4) Repeat step (3) until satisfied or exhausted to get an asymptotic series

$$S(x) = S_0(x) + S_1(x) + S_2(x)\ldots \qquad \text{as } x \to x_0$$

Dominant balance on Bessel's equation as $x \to \infty$

It is easily shown that $x_0 = \infty$ is an irregular singular point of Bessel's equation of order ν. We follow the method just described to generate asymptotic formulas for its solutions for large x.

(1) Set $y = e^{S(x)}$. Then Bessel's equation can be written

$$x^2(S'' + S'^2) + xS' + x^2 - \nu^2 = 0 \tag{17.3}$$

(2) Look for a dominant balance for large x. As discussed earlier, we neglect $S'' \ll S'^2$. In addition, $\nu^2 \ll x^2$. Thus we have

$$\underbrace{x^2 S'^2}_{A} + \underbrace{xS'}_{B} + \underbrace{x^2}_{C} \approx 0 \quad \text{for } |x| \gg 1 \tag{17.4}$$

We could solve this quadratic for S', and look for its dominant behavior at large x. However, it is easier to find a consistent dominant balance in (17.4), then solve for S'. The three choices are :

(i) $|A| \approx |B| >> |C|$

$$x^2 S'^2 + xS' \approx 0 \quad \Rightarrow \quad S' = -x^{-1} \text{ or } 0$$

Neither solution is consistent since $|C| = x^2 >> |A|, |B| = O(1)$ or 0.

(ii) $|B| \approx |C| >> |A|$

$$xS' + x^2 \approx 0 \quad \Rightarrow \quad S' = -x$$

This balance is inconsistent since $|A| = |x^2 S'^2| = x^4 >> |B|, |C| = O(x^2)$.

(iii) $|A| \approx |C| >> |B|$

$$x^2 S'^2 + x^2 \approx 0 \quad \Rightarrow \quad S' = \pm i$$

This balance is consistent since $|B| = |xS'| = O(x) << |A|, |C| = O(x^2)$.

Thus the leading-order asymptotic approximation for S' for large x is

$$S_0'^{\pm}(x) = \pm i$$

The two choices correspond to two linearly independent solutions of Bessel's equation.

(3) Set $S' = S_0' + S_r'$ and find a dominant balance for the residual S_r'. Since S_0' is the leading-order asymptotic solution, $S_r' << S_0'$ for large $|x|$.

Substitute into (17.3):

$$x^2(S_0'' + S_r'' + S_0'^2 + 2S_0'S_r' + S_r'^2) + x(S_0' + S_r') + x^2 - \nu^2 = 0 \tag{17.5}$$