Cancel out the dominant balance $x^2S_0'^2 + x^2 = 0$ in (17.5):

$$x^2(S_0'' + S_r'' + 2S_0'S_r' + S_r'^2) + x(S_0' + S_r') - \nu^2 = 0$$

To find a dominant balance, first remove terms that are obviously unimportant. Since $S_0' = \pm i$, we have $S_0'' = 0$ and $\nu^2 \ll xS_0' = \pm ix$. Since $S_r' << S_0'$, we expect $S_r'^2 \ll S_0'S_r'$. Thus

$$x^2(S_r'' + 2S_0'S_r') + xS_0' \approx 0$$

Lastly, assuming S_r' is O(x^{-a}), then $S_r'' = O(x^{-(a+1)}) \ll S_0'S_r' = \pm iS_r' = O(x^{-a})$. Thus we can neglect the first term in the above equation to obtain

$$x^2(\pm 2iS_r') \pm ix \approx 0 \quad \Rightarrow \quad S_r' \approx S_1' = -\frac{1}{2x} \tag{18.1}$$

4. *Computation of* S_2'

Take $S(x) = S_0(x) + S_1(x) + S_r(x)$, where $S_r(x)$ is a new remainder term. Substitute into (17.3):

$$x^2(S_0'' + S_1'' + S_r'' + S_0'^2 + S_1'^2 + S_r'^2 + 2S_0'S_1' + 2S_0'S_r' + 2S_1'S_r') + x(S_0' + S_1' + S_r') + x^2 - \nu^2 = 0$$

Cancel out the lower-order dominant balances $x^2S_0'^2 + x^2 = 0$ and $x^2(2S_0'S_1') + xS_0' = 0$, and note $S_0'' = 0$ to get:

$$x^2(S_1'' + S_r'' + S_1'^2 + S_r'^2 + 2S_0'S_r' + 2S_1'S_r') + x(S_1' + S_r') - \nu^2 = 0$$

Neglect terms which cannot be dominant - $S_r'^2 \ll S_1'^2, 2S_1'S_r' \ll 2S_0'S_r', S_r'' \ll S_1'', S_r' \ll S_1'$:

$$x^2(S_1'' + S_1'^2 + 2S_0'S_r') + xS_1' - \nu^2 \approx 0$$

Substitute $S_0' = \pm i$ and $S_1' = -1/2x$:

$$x^2(\frac{1}{2x^2} + \frac{1}{4x^2} \pm 2iS_r') + x\left(-\frac{1}{2x}\right) - \nu^2 \approx 0$$

All the known terms are O(1), so none can be neglected. We solve this equation for an asymptotic approximation S_2' to S_r':

$$\pm 2ix^2S_2' = \nu^2 - \frac{1}{4} \quad \Rightarrow \quad S_2' = \pm\frac{i}{2x^2}\left\{\frac{1}{4} - \nu^2\right\} \tag{18.2}$$

We could clearly continue this process to develop an asymptotic series in which $S_n' = O(x^{-n})$. Putting the terms we have together,

$$S'^{\pm}(x) = \pm i - \frac{1}{2x} \pm \frac{i}{2x^2}\left\{\frac{1}{4} - \nu^2\right\}\ldots$$

Integrating (ignoring integration constants which are additive constants in $S(x)$ and hence multiplicative constants in e^S, and so do not lead to new solutions) we find:

$$S^{\pm}(x) = \pm ix - \frac{1}{2}\log x \pm \frac{i}{2x}\left\{\nu^2 - \frac{1}{4}\right\}\ldots$$

$$y^{\pm}(x) = \exp[S^{\pm}(x)] = \exp\left[\pm ix - \frac{1}{2}\log x \pm \frac{i}{2x}\left\{\nu^2 - \frac{1}{4}\right\}\ldots\right]$$

$$= x^{-1/2}\exp\left(\pm\left[ix + \frac{i}{2x}\left\{\nu^2 - \frac{1}{4}\right\}\ldots\right]\right)$$

These solutions are (to within constants) asymptotic series for the Hankel functions of order n:

$$H_\nu^{(1,2)}(x) = J_\nu(x) \pm iY_\nu(x) = \left(\frac{2}{\pi}\right)^{1/2} e^{\mp i(\nu\pi/2+\pi/4)} y^{\pm}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} e^{\pm i(x-\nu\pi/2-\pi/4+(\nu^2-1/4)/2x\ldots)}$$

We can get corresponding series for $J_\nu(x)$ and $Y_\nu(x)$ by taking real and imaginary parts of the above.

The plot below, made by Matlab script `bessel_large_x.m` (see class web page) shows the $\exp(S_0+S_1)$ approximations to $J_0(x)$ and $Y_0(x)$ are amazingly accurate down to $x = 2.5$, and adding the S_2 correction term makes them accurate down to $x = 1.5$. Thus one could use the first three terms of the Frobenius series for $|x| < 1.5$ and the first three terms of the large-x asymptotic series for $|x| > 1.5$ to get an excellent, computationally efficient global approximation to $J_0(x)$ and $Y_0(x)$.