

An asymptotic series which does not converge

The equation $y'' + \frac{2}{x}y' = 0$ has an irregular singular point at $x = \infty$. One can show that applying dominant balance leads to one solution which is exactly constant and a second solution in a series in the form

$$y(x) = ce^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1 \cdot 3}{2^3x^5} - \frac{1 \cdot 3 \cdot 5}{2^4x^7} \dots \right\}$$

Actually it is not to easy to deduce the general pattern from the method of dominant balance, but in this case if $c = 2/\pi^{1/2}$, this solution corresponds to the exact solution $y = \operatorname{erfc}(x)$, which can be repeatedly integrated by parts to derive a series of the same form, which therefore (wave hands at the mathematicians here!) must be the same series:

$$\begin{aligned} \operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = \frac{2}{\sqrt{\pi}} \left\{ \left[-\frac{1}{2u} e^{-u^2} \right]_x^\infty - \int_x^\infty \frac{e^{-u^2}}{2u^2} du \right\} \\ &= \frac{2}{\sqrt{\pi}} \left\{ \frac{1}{2x} e^{-x^2} + \left[\frac{1}{2^2u^3} e^{-u^2} \right]_x^\infty + \int_x^\infty \frac{3}{2^2} u^{-4} e^{-u^2} \dots \right\} \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1 \cdot 3}{2^3x^5} - \frac{1 \cdot 3 \cdot 5}{2^4x^7} \dots \right\} \end{aligned}$$

As $x \rightarrow \infty$, the error in this series truncated after N terms is approximately equal to the $N+1$ 'st term, which decays like an increasingly high power $x^{-(2N+1)}$ of x , so the series is an asymptotic series. However, at *fixed* x , the error in the series starts becoming *larger* as N increases when $N > x^2$, since the magnitude of term $N+1$ is $(2N-1)/(2x^2)$ times that of term N , so when $N > x^2$ the terms increase in magnitude as N increases.

Recall the properties of an asymptotic series $\sum_{n=0}^{\infty} t_n(x)$ for a function $y(x)$ as $x \rightarrow x_0$. Let

$s_P(x) = \sum_{n=0}^P t_n(x)$ be the P th partial sum, and let $\epsilon_P(x) = y(x) - s_P(x)$ be its residual error. Then

- (1) $y(x) \sim$ (“is asymptotic to”) $s_0(x)$ as $x \rightarrow x_0$, i. e., the ratio $s_0(x)/y(x) \rightarrow 1$ as $x \rightarrow x_0$. That is, the leading term in the series captures the leading asymptotic behavior of $y(x)$.
- (2) For any P , the error $\epsilon_P(x) \sim t_{P+1}(x)$ as $x \rightarrow x_0$. That is, each term in the series captures the leading asymptotic behavior of the error of the previous partial sum of the series.
- (3) The series may or may not converge for any fixed x as $P \rightarrow \infty$. If it does not, there will be an optimal number of terms $N(x)$ which yields the most accurate approximation to $y(x)$. In general, $N(x)$ increases and a more accurate approximation can be found as $x \rightarrow x_0$.

The figure below (generated by Matlab script `erfc_largex.m` on class web page) illustrates this behavior on the asymptotic series we just derived for $\text{erfc}(x)$ as $x \rightarrow \infty$.

