Regular Perturbation Series

The solution of a regular perturbation problem is usually a smooth function of some power q of ε (e. g. ε, ε^2 or $\varepsilon^{1/2}$) near ε = 0. In this case, we can expand the solution in a Taylor series in ε^q, which we call a *perturbation series*:

$$y(x,\varepsilon) = y_0(x) + \varepsilon^q y_1(x) + \varepsilon^{2q} y_2(x)\ldots \tag{2.1}$$

The function $y_0(x)$ (the solution of the ε = 0 problem) and the unknown functions $y_1(x)$, $y_2(x)$, … are found by substituting (2.1) into the original problem (including any boundary conditions), separating the terms according to powers of ε^q, and satisfying the successive problems at each order in ε. The exponent q is usually 1, but when the choice of q is unclear, it can be made by substituting the proposed solution into the original equation and seeing what q is required to balance the perturbation terms (the higher powers of ε) in the equation. Like a normal Taylor series, the perturbation series is unique; if a series in powers of ε is assumed for $y(x, \varepsilon)$, but $y(x, \varepsilon)$ could actually be written as a series in powers of ε^2, then the coefficients of the odd powers of will all turn out to be identically zero. However, if $y(x, \varepsilon)$ really were a series in powers of $\varepsilon^{1/2}$, then our assumed series in powers of ε would be inconsistent. That is, we would find some orders of ε gave unbalanced equations where a nonzero function must equal zero. This would indicate that q should be changed to remove these imbalances.

<u>Example (R1)</u>

Find perturbation series for the roots of $p(r;\varepsilon) = r^2 + \varepsilon r - 1 = 0, \quad \varepsilon \ll 1$

Note that for this problem, the two roots can be found exactly for any ε using the quadratic formula:

$$r^{\pm}(\varepsilon) = \frac{-\varepsilon \pm (4+\varepsilon^2)^{1/2}}{2} \tag{2.2}$$

so this is just a pedagogical exercise to get us started. We can see that the two roots are smooth functions of ε for small ε and we can anticipate from (2.2) that the perturbation series will be the Taylor series in ε that we could derive from (2.2), and that because $r^{\pm}(\varepsilon)$ are analytic functions of the complex variable ε out to the nearest singularities (branch points at ε = ±2i), these series will have a radius of convergence of 2, i. e. they will converge for |ε| < 2. Let's see if this really works!

Let us try a perturbation series of the form

$$r(\varepsilon) = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots$$

and substitute into the polynomial:

$$0 = r^2 + \varepsilon r - 1 = (r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots)^2 + \varepsilon(r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots) - 1$$
$$= (r_0^2 - 1) + \varepsilon(2r_0 r_1 + r_0) + \varepsilon^2(2r_0 r_2 + r_1^2 + r_1)\ldots$$

For this to hold for arbitrary small values of ε, coefficients of all powers of ε must vanish:

ε^0: $\quad 0 = r_0^2 - 1 \quad \Rightarrow \quad r_0^{\pm} = \pm 1$

ε^1: $\quad 0 = 2r_0 r_1 + r_0 \quad \Rightarrow \quad r_1^{\pm} = -\dfrac{1}{2}$

ε^2: $\quad 0 = 2r_0 r_2 + r_1^2 + r_1 \quad \Rightarrow \quad r_2^{\pm} = -\dfrac{r_1^2 + r_1}{2r_0} = -\dfrac{-1/4}{\pm 2} = \pm\dfrac{1}{8}$

so we obtain the perturbation series for the two roots:

$$r^+(\varepsilon) = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 \ldots$$

$$r^-(\varepsilon) = -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 \ldots$$

The series for $r^+(\varepsilon)$ is indeed the same as the Taylor series of the exact solution around 0:

$$\frac{-\varepsilon + (4 + \varepsilon^2)^{1/2}}{2} = \frac{-\varepsilon}{2} + \left(1 + \frac{\varepsilon^2}{4}\right)^{1/2} = \frac{-\varepsilon}{2} + 1 + \frac{\varepsilon^2}{8}\ldots$$

and similarly for $r^-(\varepsilon)$.

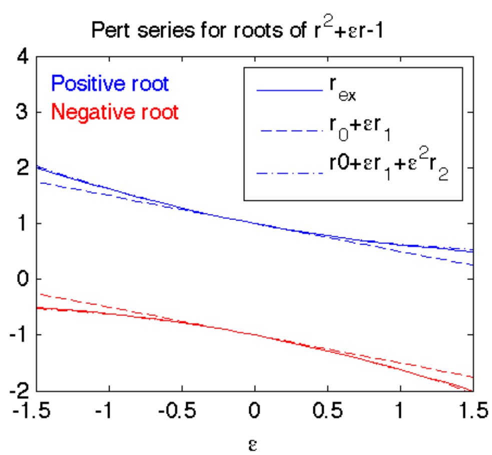

Fig. 2.1: Convergence of perturbation series for (R1)

In this case, the perturbation series are quite accurate even for fairly large values of ε (Fig. 2.1). If only the O(ε)) term is kept in the series for either root, the $O(\varepsilon^2)$ term gives an error estimate of roughly $\varepsilon^2/8$. If the $O(\varepsilon^2)$ term is kept in the perturbation series, it remains highly accurate out to |ε| ≈ 1.5.

<u>A trickier root-finding example</u>

We wish to find perturbation series for the roots of the cubic equation

$$p(r;\varepsilon) = r^3 + r^2 - \varepsilon = 0, \quad \varepsilon \ll 1. \tag{2.3}$$

Its real roots (but not any complex roots) can be visualized by plotting *p*(*r*; ε) as in Fig. 2.2; the roots are where it crosses the *r*-axis. From the graph, we can see that the unperturbed problem has a root $r_0^I = -1$ and a double root $r_0^{II,III} = 0$ (we can tell it is a double root because *dp/dr* is also zero at *r* = 0). The double root of the unperturbed problem is what makes the perturbation series more tricky. Graphically, we can see that for a small positive ε, *p*(*r*; ε) locally looks like an upward facing parabola around its minimum value -ε at *r* = 0. Thus we can anticipate that the two roots $r^{II,\,III}(\varepsilon)$ will be roughly $\pm O(\varepsilon^{1/2})$, rather than being linear functions of ε. On the other hand, the simple root $r^I(\varepsilon)$ will be -1+O(ε). Let's explore how this emerges from the perturbation series approach!

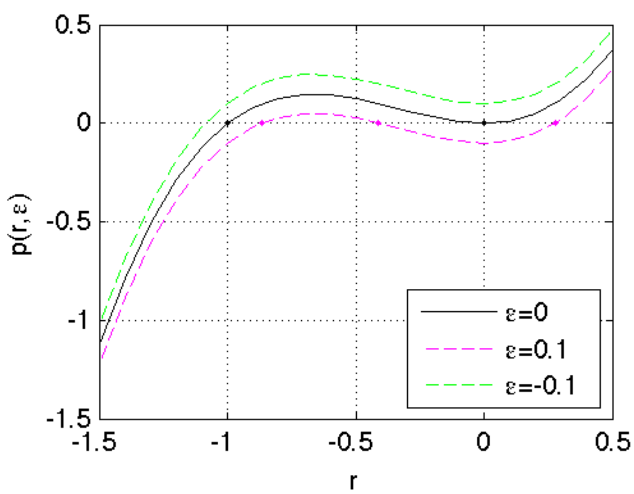

Fig. 2.2: Graph of *p*(*r*; ε); roots marked as dots for ε = 0 and 0.1.