

Two-scale analysis of dispersive wave packets

Linear 1D waves in a homogeneous medium are characterized by a *dispersion relation* $\omega(k)$ which relates their frequency ω to their phase speed k . A *monochromatic* wave of a single wavenumber k has the form $a_0 \exp(i[kx - \omega t])$, where a_0 is a constant amplitude. The wave phase $\phi(x, t) = kx - \omega t$ propagates to the right at the *phase speed* $c = \omega(k)/k$.

Linear waves are called *dispersive* if the phase speed varies with wavenumber k . Sound or light waves are nondispersive, i. e. all wavelengths have the same phase speed. However, for rightward propagating surface gravity waves on a layer of water of depth h , $\omega(k) = [gk \tanh(kH)]^{1/2}$. Small wavenumbers (long wavelengths) with $kH \ll 1$ (e. g. tsunami waves) have phase speed $(gH)^{1/2}$, while large wavenumbers (short wavelengths) with $kH \gg 1$ (surface ripples) have slower phase speed $(g/k)^{1/2}$.

A one-dimensional *wave packet* is a spatially localized wave disturbance with some characteristic *carrier wavenumber* k_0 and frequency ω_0 modulated by a possibly complex amplitude $a(x, t)$:

$$y(x, t) = a(x, t) \exp(ik_0 x - i\omega_0 t)$$

For instance, think of a tsunami produced by a localized earthquake propagating across the ocean. We wish to predict how the wave packet evolves, i. e. to find a partial differential equation for the evolution of $a(x, t)$, given $a(x, 0)$ and the dispersion relation governing the waves. Because of the amplitude modulation, y is a superposition of many Fourier wavenumbers k . For instance, if $a(x, 0) = a_0 \exp(-x^2 / 2\sigma^2)$, the Fourier transform of $y(x, 0)$ is

$$\begin{aligned} \hat{y}(k, 0) &= \int_{-\infty}^{\infty} y(x, 0) \exp(-ikx) dx = \int_{-\infty}^{\infty} a_0 \exp(-x^2 / 2\sigma^2) \exp(i[k_0 - k]x) dx \\ &= a_0 \exp(-\sigma^2 [k_0 - k]^2 / 2) \int_{-\infty}^{\infty} \exp(-\{x - i\sigma^2 [k_0 - k]\}^2 / 2\sigma^2) dx \\ &= a_0 \sigma (2\pi)^{1/2} \exp(-\sigma^2 [k_0 - k]^2 / 2) \end{aligned}$$

Thus, if the wave packet has a characteristic width $\delta x = \sigma$ in physical space, it has a characteristic width $\delta k = \sigma^{-1}$ around the carrier wavenumber k_0 in Fourier space.

If the wave packet is narrow in Fourier space (broad in physical space), a Taylor series approximation to the dispersion relation around k_0

$$\omega(k) = \omega_0 + (k - k_0) \frac{\partial \omega}{\partial k}(k_0) + \frac{(k - k_0)^2}{2} \frac{\partial^2 \omega}{\partial k^2}(k_0) \dots$$

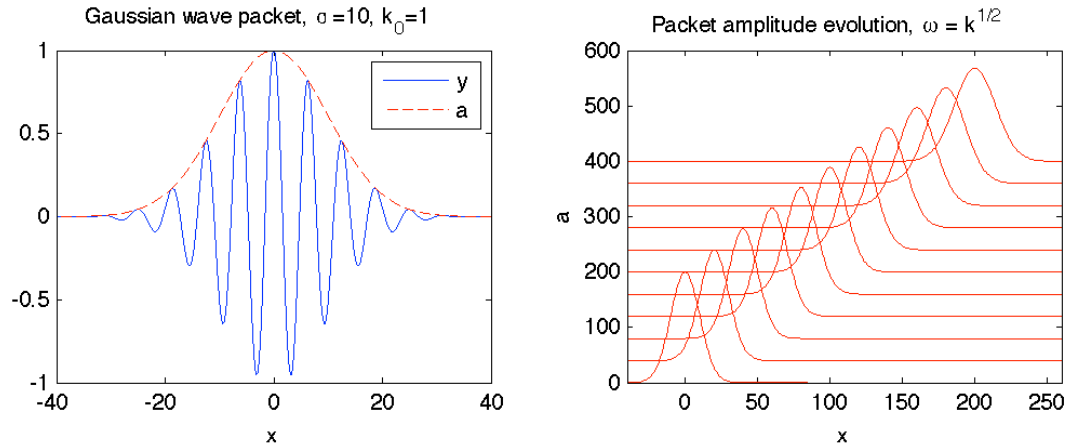
will be accurate across the range of wavenumbers that make up most of the wave packet. This greatly simplifies the calculation of the wave packet evolution. For instance, for a Gaussian wave packet with $\delta k \ll k_0$,

$$\begin{aligned}\hat{y}(k,t) &= \hat{y}(k,0)\exp\{-i\omega(k)t\} \\ &= a_0\sigma(2\pi)^{1/2}\exp(-\sigma^2[k_0 - k]^2/2)\exp\left\{-i\left[\omega_0 + (k - k_0)\frac{\partial\omega}{\partial k}(k_0) + \frac{(k - k_0)^2}{2}\frac{\partial^2\omega}{\partial k^2}(k_0)\dots\right]t\right\} \\ &= a_0\sigma(2\pi)^{1/2}\exp(-i\omega_0 t)\exp\left\{-i(k - k_0)\frac{\partial\omega}{\partial k}(k_0)t - \frac{(k - k_0)^2}{2}\left(\sigma^2 + i\frac{\partial^2\omega}{\partial k^2}(k_0)t\right)\dots\right\}\end{aligned}$$

Defining the *group velocity* $c_g(k) = \partial\omega / \partial k$ and the *dispersion* $\beta(k) = \partial^2\omega / \partial k^2$, using a subscript 0 to denote their value at the carrier wavenumber k_0 , and dropping the ‘...’ terms, we see that

$$\begin{aligned}\hat{y}(k,t) &\approx a_0\sigma(2\pi)^{1/2}\exp(-i\omega_0 t)\exp\left\{-i(k - k_0)c_{g0}t - \frac{(k - k_0)^2}{2}(\sigma^2 + i\beta_0 t)\right\} \\ y(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{y}(k,t)\exp(ikx)dk \\ &\approx \frac{a_0\sigma\exp(-i\omega_0 t)}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\exp\left\{ikx - i(k - k_0)c_{g0}t - \frac{(k - k_0)^2}{2}(\sigma^2 + i\beta_0 t)\dots\right\}dk \\ &= \frac{a_0\sigma\exp(i[k_0 x - \omega_0 t])}{(2\pi)^{1/2}}\underbrace{\int_{-\infty}^{\infty}\exp\left\{ik_1(x - c_{g0}t) - \frac{k_1^2}{2}(\sigma^2 + i\beta_0 t)\right\}dk_1}_{k_1 = k - k_0} \\ &= \frac{a_0\sigma\exp(i[k_0 x - \omega_0 t])}{(2\pi)^{1/2}}\exp\left\{-\frac{(x - c_{g0}t)^2}{2(\sigma^2 + i\beta_0 t)}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{(\sigma^2 + i\beta_0 t)}{2}\left[k_1 - i\frac{(x - c_{g0}t)}{(\sigma^2 + i\beta_0 t)}\right]^2\right\}dk_1 \\ &= a_0\exp(i[k_0 x - \omega_0 t])\frac{\sigma}{(\sigma^2 + i\beta_0 t)^{1/2}}\exp\left\{-\frac{(x - c_{g0}t)^2}{2(\sigma^2 + i\beta_0 t)}\right\} \\ \Rightarrow a(x,t) &= \frac{a_0}{(1 + i\beta_0 t / \sigma^2)^{1/2}}\exp\left\{-\frac{(x - c_{g0}t)^2}{2(\sigma^2 + i\beta_0 t)}\right\}\end{aligned}$$

The Gaussian packet moves at the carrier group velocity c_{g0} and it slowly spreads out due to the dispersion β_0 . Fig. 1 shows the example $\omega = k^{1/2}$ (deep water waves) with $\sigma = 10$, $k_0 = 1$, $c_{g0} = 0.5$, $\beta_0 = -0.25$. The left panel shows the wave packet at $t = 0$, and the right panel shows how the absolute value of the amplitude evolves; the dispersion is a rather weak effect, broadening the packet about 20% in the 400 time units plotted.



You have probably been taught that these aspects of wave packet dynamics apply generally...but how can we show this mathematically with the tools we have learned? The trick is to introduce a small parameter

$$\varepsilon = \delta k / k_0 \ll 1$$

which characterizes the wave packet width in Fourier space. The characteristic packet width in physical space is $\delta x = \varepsilon^{-1} / k_0$, which is long compared to the lengthscale $1 / k_0$ on which the carrier wave varies. We use this to define long space and time scales over which the packet amplitude evolves.

We use a form of multiple scale analysis. We introduce a long length coordinate

$$X = \varepsilon x$$

Thus the packet amplitude varies on a lengthscale $\delta X = \varepsilon^{-1} \delta x = 1 / k_0$, while the carrier wave varies on a lengthscale $\delta x = 1 / k_0$. Since all the frequencies in the packet differ from the carrier frequency by $O(\varepsilon)$, we can also anticipate that the packet amplitude will evolve on a timescale $\varepsilon^{-1} / \omega_0$. Hence we also define a long timescale $T = \varepsilon t$ and we write $a(x, t) = A(X, T)$. The wave packet has the form

$$y(x, t) = A(X, T) \exp(i[k_0 x - \omega_0 t])$$

We assume that we know the initial packet shape $A(X, 0)$, such that $y(x, 0) = A(X, 0) \exp(ik_0 x)$; our task is to derive an equation for $A(X, T)$. We also define a scaled wavenumber $K = \varepsilon^{-1}(k - k_0)$.

The Fourier transform of the initial wave packet is

$$\begin{aligned} \hat{y}(k, 0) &= \int_{-\infty}^{\infty} y(x, 0) \exp(-ikx) dx = \int_{-\infty}^{\infty} A(X, 0) \exp(i[k_0 - k]x) dx \\ &= \varepsilon^{-1} \int_{-\infty}^{\infty} A(X, 0) \exp(-iKX) dX = \varepsilon^{-1} \hat{A}(K, 0) \end{aligned}$$

Each wavenumber k evolves as $\exp(-i\omega(k)t)$, so

$$\begin{aligned}\hat{y}(k,t) &= \exp(-i\omega(k)t)\hat{y}(k,0) = \exp\left\{-i\left[\omega_0 + \varepsilon K \frac{\partial\omega}{\partial k}(k_0) + \varepsilon^2 \frac{K^2}{2} \frac{\partial^2\omega}{\partial k^2}(k_0) \dots\right]t\right\} \varepsilon^{-1} \hat{A}(k,0) \\ &= \varepsilon^{-1} \exp(-i\omega_0 t) \hat{A}(K,T),\end{aligned}$$

where
$$\hat{A}(K,T) = \exp\left\{-i\left[Kc_{g0} + \varepsilon \frac{K^2}{2} \beta_0 \dots\right]T\right\} \hat{A}(K,0) \quad (20.1)$$

Introducing the small parameter ε allows us to formally justify neglecting higher-order terms in the Taylor series, and also shows the dispersion is a higher order effect than the group velocity.

We deduce that $\hat{A}(K,T)$ satisfies the ODE

$$\frac{\partial \hat{A}}{\partial T} = -i \left[Kc_{g0} + \varepsilon \frac{K^2}{2} \beta_0 \dots \right] \hat{A}$$

To take the inverse Fourier transform, note that $iK \rightarrow \partial/\partial x$, yielding the desired PDE

$$\frac{\partial A}{\partial T} \approx -c_{g0} \frac{\partial A}{\partial X} + \frac{i\beta_0 \varepsilon}{2} \frac{\partial^2 A}{\partial X^2} \quad (20.2)$$

for the wave packet amplitude. To leading order, this is a one-way wave equation for A , whose solution is that $A(X, T) = A(X - c_{g0}t, 0)$, a wave propagating at the group velocity c_{g0} . At $O(\varepsilon)$ there is a correction due to the dispersion. This correction only becomes significant when $T = O(\varepsilon^{-1})$ or $t = O(\varepsilon^{-2})$. The Gaussian solution derived earlier satisfies this PDE; for that case $\varepsilon = (\sigma k_0)^{-1}$ ($= 0.1$ in the plotted example).