

Large- x asymptotics of the Airy function using method of stationary phase

Recall the Airy function $\text{Ai}(x)$, which is the solution of Airy's equation

$$y'' - xy = 0 \quad (21.1)$$

that goes to zero as $x \rightarrow \infty$. In lecture 12, we quoted asymptotic formulas for $\text{Ai}(x)$ for large positive and negative x . We could derive the general form of these behaviors using dominant balance for large x (since ∞ is an irregular singular point), but these do not connect the behavior between positive and negative x . To do this, we apply another elegant approach, in which we use a Fourier transform of (21.1) to express $\text{Ai}(x)$ as an integral which we can approximate for large negative x using **the method of stationary phase** and for large positive x using a generalization called the **method of steepest descents**.

The Fourier transform of (21.1) is found by multiplying by e^{-ikx} and integrating wrt x :

$$0 = \int_{-\infty}^{\infty} (y'' - xy) e^{-ikx} dx \quad (21.2)$$

We look for a solution for which $y(x) \rightarrow 0$ as $x \rightarrow \pm\infty$; this eliminates the second linearly independent solution to Airy's equation, leaving only $\text{Ai}(x)$. In this case we can integrate the first term twice by parts to get $-k^2 \hat{y}(k)$. A trick allows the second term to be simplified:

$$\frac{d\hat{y}}{dk} = \frac{d}{dk} \int_{-\infty}^{\infty} y(x) e^{-ikx} dx = \int_{-\infty}^{\infty} y(x) \frac{d}{dk} e^{-ikx} dx = -i \int_{-\infty}^{\infty} xy e^{-ikx} dx \quad (21.3)$$

so (21.2) can be written

$$0 = -k^2 \hat{y} + \frac{1}{i} \frac{d\hat{y}}{dk} \Rightarrow \frac{d\hat{y}}{dk} = ik^2 \hat{y} \Rightarrow \hat{y}(k) = c \exp\left(\frac{ik^3}{3}\right)$$

The conventional normalization of the Airy function is obtained by taking $c = 1$. Taking the inverse transform,

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(\frac{ik^3}{3} + ikx\right) dk \quad (21.4)$$

This is a rather strange integral; the integrand does not go to zero as $x \rightarrow \pm\infty$ yet the integral still converges because the complex exponential oscillate faster and faster as $|k|$ increases, producing increasingly complete cancellation of the contributions between its different phases. We can take advantage of this behavior to approximately evaluate the integral for large positive and negative x . The integral is real, so for visualization Fig. 1 plots the real part of the integrand, i. e. $\cos(\text{phase})$.

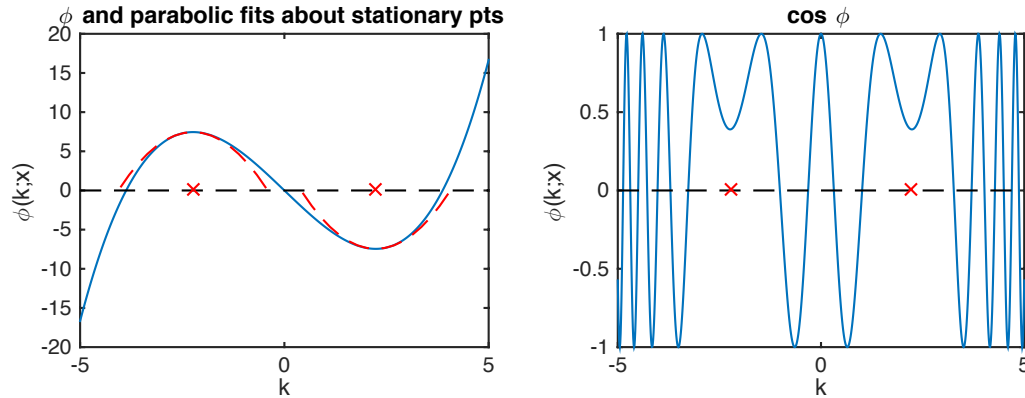


Fig. 1: Phase (left) and integrand (right) for the Airy Fourier transform. Red crosses are stationary phase points. Red dashes in left plot show parabolic fits to phase around stationary phase points.

In general, consider an integral of the type

$$I(x) = \int_{-\infty}^{\infty} f(k) e^{i\phi(k;x)} dk$$

We anticipate the integral will be dominated by those k for which the phase $\phi(k; x)$ is not a rapidly changing function of k , because for other k , adjacent positive and negative contributions will mostly cancel. Thus, we

- (1) Look for $k_0(x)$ at which the phase is stationary (doesn't depend on k), i. e. $\partial\phi/\partial k = 0$.
- (2) Locally approximate:

$$\phi(k; x) = \phi_0 + \underbrace{\phi_0'}_0 (k - k_0) + \frac{\phi_0''}{2} (k - k_0)^2 + O[\phi_0'''(k - k_0)^3]$$

$$f(k) = f_0 + f_0'(k - k_0) + \frac{f_0''}{2} (k - k_0)^2 + O[f_0'''(k - k_0)^3]$$

Here, a subscript 0 implies evaluation at $k = k_0$.

- (3) Substitute these approximations into the integral, keeping the minimum set of terms necessary to get a convergent integral:

$$I(x) \approx \int_{-\infty}^{\infty} (f_0 + \dots) \exp\left(i\phi_0 + i\frac{\phi_0''}{2}[k - k_0]^2\right) dk = f_0 e^{i\phi_0} \int_{-\infty}^{\infty} \exp\left(i\frac{\phi_0''}{2}[k - k_0]^2\right) dk$$

The **Fresnel integral** is calculated by letting $a = \phi_0''$. If $a > 0$, set $z = (k - k_0) e^{-i\pi/4}$:

$$\int_{-\infty}^{\infty} \exp\left(i\frac{a}{2}[k - k_0]^2\right) dk = e^{i\pi/4} \int_{-\infty e^{-i\pi/4}}^{\infty e^{-i\pi/4}} \exp\left(-\frac{a}{2}z^2\right) dz$$

The contour can be rotated to the real axis by appealing to Jordan's Lemma, so

$$e^{i\pi/4} \int_{-\infty e^{-i\pi/4}}^{\infty e^{-i\pi/4}} \exp\left(-\frac{a}{2} z^2\right) dz = e^{i\pi/4} \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2} z^2\right) dz = e^{i\pi/4} \left(\frac{2\pi}{a}\right)^{1/2}$$

Similarly, if $a < 0$,

$$\int_{-\infty}^{\infty} \exp\left(i\frac{a}{2}[k-k_0]^2\right) dk = e^{-i\pi/4} \int_{-\infty e^{i\pi/4}}^{\infty e^{i\pi/4}} \exp\left(-\frac{|a|}{2} z^2\right) dz = e^{-i\pi/4} \left(\frac{2\pi}{|a|}\right)^{1/2}$$

Putting this all together,

$$I(x) = f_0 e^{i(\phi_0 + \pi \operatorname{sgn}(a)/4)} \left(2\pi/|\phi_0''|\right)^{1/2}$$

Note that the range in k that significantly contributes to this integral will be:

$$\phi_0''[k-k_0]^2 = O(1) \Rightarrow |k-k_0| = O(|\phi_0''|^{-1/2})$$

If there are multiple stationary phase points, sum their contributions to the integral.

- (4) The minimal approximation to the integrand will be accurate in the region $|k-k_0| \ll \min(1, |\phi_0''/\phi_0'''|)$, in which $f(k) \approx f_0$ and the cubic term in the Taylor series for $\phi(k; x)$ is negligible compared to the retained quadratic term. For this to hold over the entire range of k that contributes significantly to the integral $I(x)$, x is restricted to cases in which

$$|\phi_0''|^{-1/2} \ll \min(1, |\phi_0''/\phi_0'''|) \Rightarrow |\phi_0''| \gg 1, |\phi_0'''|^{2/3}$$

Application to Airy Fourier transform integral (21.4)

For this case, $f(k) = (2\pi)^{-1/2}$ and

$$\phi(k, x) = \frac{k^3}{3} + kx$$

Thus, $f_0 = (2\pi)^{-1/2}$ (no local approximation to $f(k)$ required in this case) and the stationary phase wavenumbers k_0 are found from

$$0 = \frac{\partial \phi}{\partial k} = k^2 + x \Rightarrow k_0 = \pm(-x)^{1/2}$$

We immediately see this approach will work for negative x only. For that case,

$$\phi_0 = \frac{k_0^3}{3} + k_0 x = -\frac{2}{3} k_0^3, \quad \phi_0'' = 2k_0, \quad \phi_0''' = 2$$

Thus, the approximate value of the integral is the sum of contributions from each of the two stationary-phase k_0 's, each of the form

$$f_0 e^{i(\phi_0 + \pi/4)} (2\pi/\phi_0'')^{1/2} = \frac{1}{2\pi} \left(\frac{2\pi}{2k_0} \right)^{1/2} \exp i \left(-\frac{2}{3} k_0^3 + \frac{\pi}{4} \right)$$

Summing these contributions, we get the approximation given in Lecture 12:

$$\text{Ai}(x) = \left(\frac{1}{4\pi|x|^{1/2}} \right)^{1/2} \left\{ \underbrace{\exp i \left(-\frac{2}{3}|x|^{3/2} + \frac{\pi}{4} \right)}_{k_0=|x|^{1/2}} + \underbrace{\exp i \left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} \right)}_{k_0=-|x|^{1/2}} \right\}$$

$$\text{Ai}(x) \approx \pi^{-1/2} |x|^{-1/4} \cos \left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} \right) = \pi^{-1/2} |x|^{-1/4} \sin \left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4} \right)$$

The approximation will be accurate if

$$|\phi_0''| \gg |\phi_0'''|^{2/3} \Rightarrow 2|k_0| \gg 4^{2/3} \Rightarrow |x| \gg 1$$